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Classifying Graph Lie Algebras

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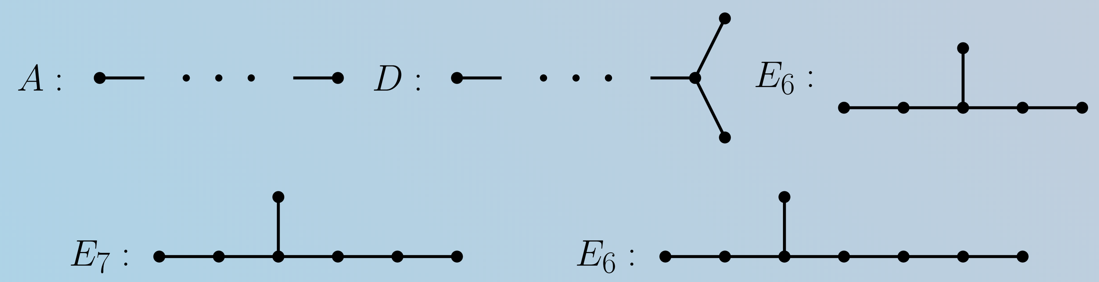
Introduction

A Lie group is a structure concerned with infinitesimal transformations, introduced by Sophus Lie in 1876 [2]. It is used for analyzing continuous symmetries of mathematical objects, giving it applications in differential geometry as well as theoretical physics. However, Lie groups can be studied more simply by considering the Lie algebras they give rise to. A Lie algebra is a linear object defined by a Lie group; every Lie algebra is locally defined by the Lie group which gives rise to it, but Lie algebras often are easier to study as a large amount of work has been done on linear objects.



A paper by Khovanova introduces and studies Lie algebras corresponding to Dynkin diagrams of simple graphs, seen in the diagram above. In this paper we extend this work further by building and classifying Lie algebra structures defined by any simple graph.

Building a Graph Lie Algebra

A previous paper by Khovanova [3] defines the *graph algebra* $\mathcal{A}(G)$ of a graph G with n vertices $v_1, \ldots, v_n$ as a unital algebra over $\mathbb{C}$ with n generators $e_1, \ldots, e_n$, called *vertex monomials*, with the following properties:

- For all $1 \le i \le n$, we have $e_i^2 = -1$.
- For all $i \neq j$, we have that $e_i e_j = -e_j e_i$ if v_i and v_j are connected, and $e_i e_j = e_j e_i$ otherwise.

A Lie algebra is not an algebra. Rather, it is a vector space L equipped with addition and a bilinear operator $[L, L] \to L$, called the *Lie bracket*, which follows the *Jacobi identity* for all $x, y, z \in L$:

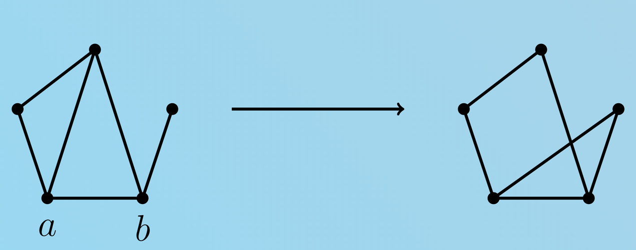
$$[x, y] + [y, z] + [z, x] = 0.$$

The *graph Lie algebra* $\mathfrak{L}(G)$ of a graph G is the smallest Lie algebra which is a subspace of the vector space of $\mathcal{A}(G)$ and contains every vertex monomial, when the Lie bracket is defined as follows:

$$[x, y] = xy - yx.$$

Manipulating Graphs

We define a *flip* between two vertices as toggling whether they are connected: adding an edge if none exists, and removing an edge if one is present. Then a *swap* about vertex a with respect to a vertex b is the act of flipping between a and every vertex connected to b, not including a, as shown below:



We also define a *connected swap* as a swap about some vertex with respect to a vertex it is connected to. We can prove a nice theorem connecting this alteration to our graph Lie algebras:

Theorem 1. *When a connected swap is performed on a simple graph, the graph Lie algebra of the resulting graph is isomorphic to the graph Lie algebra of the original.*

This theorem allows us to alter our graph while preserving the underlying graph Lie algebra.

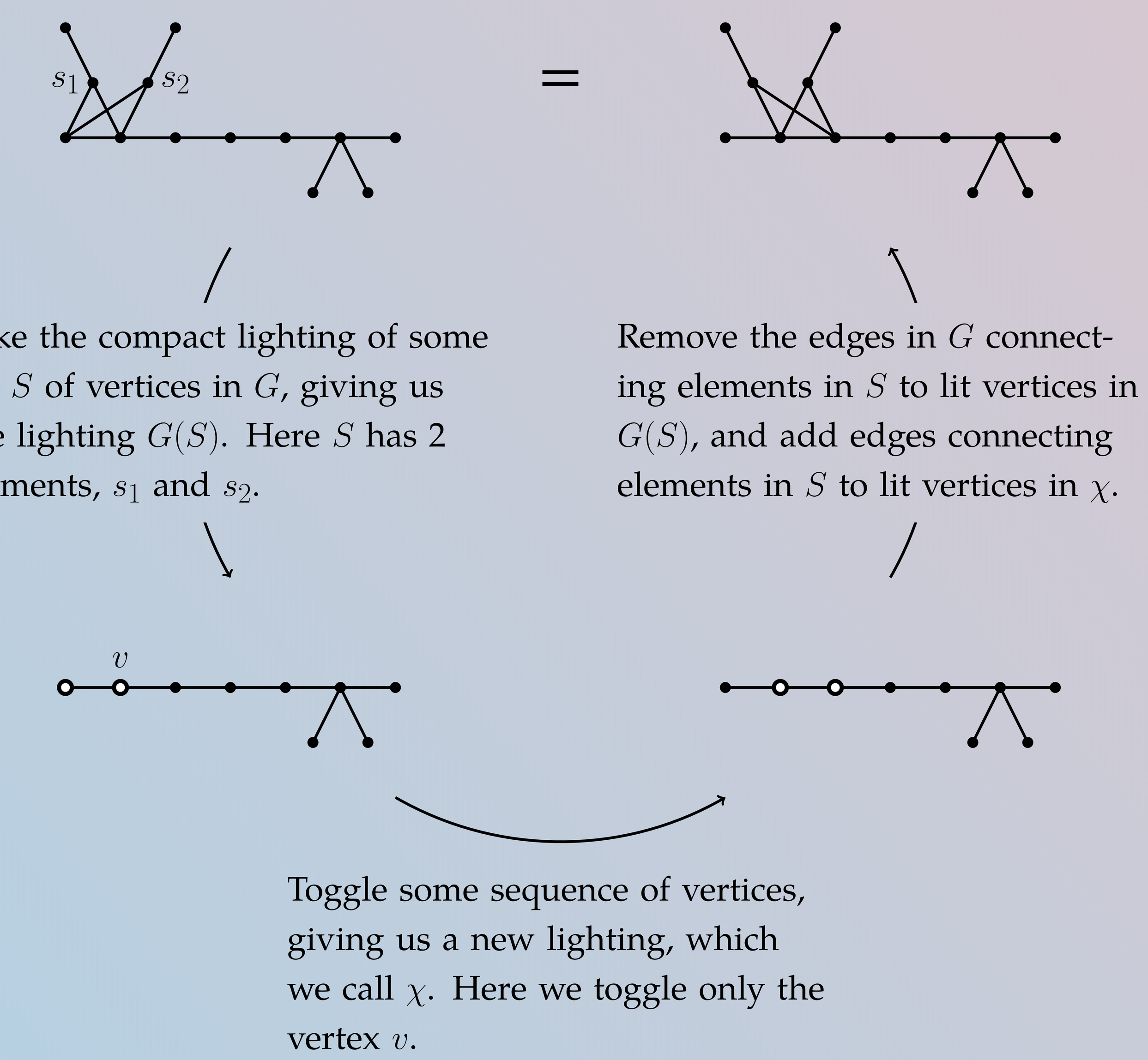
Condensing Swaps

Once swaps are proven to preserve graph Lie algebras, we can use them to manipulate the general simple graph. However, the sequences of swaps used to simplify graphs are very long and hard to manage, so in order to perform them we must derive new ways of performing large numbers of swaps in a more comprehensible way. To do so we introduce the *lit-only sigma game*, as seen in [1]:

Lemma 1. *Any endgame of any simple graph G has a graph Lie algebra isomorphic to that of G.*

Proof. Say we take the compact lighting of a set S, then toggle about v, then take the endgame of the resulting lighting. This is equivalent to swapping about every vertex in S with respect to v. □

Using this lemma, given a simple graph G, we can find a graph with an isomorphic graph Lie algebra using the lit-only sigma game, as seen in the diagram below.

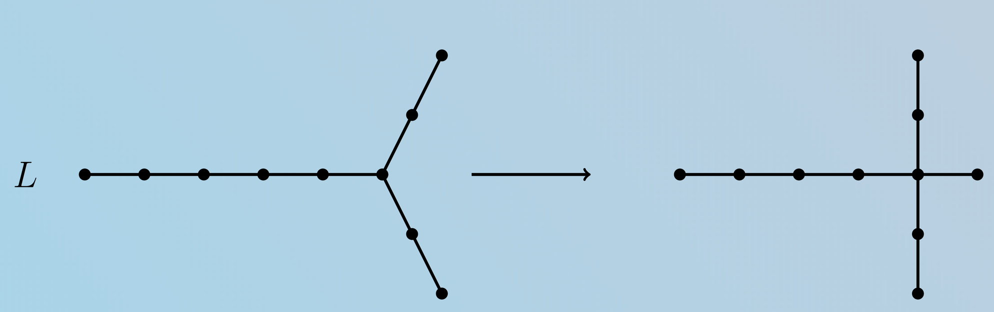


Define a *extended star* as a tree where at most one vertex has degree greater than 2. Additionally, every connected component resulting when all vertices of degree greater than 2 are path graphs called *legs*. A *minimal extended star* is a graph that is either a path graph or an extended star with at least 3 legs where the longest distance between the vertex with degree greater than 2 and any other vertex is at most 4. Using the lit-only sigma game to help simplify large sequences of swaps, we can use swaps to simplify all graph Lie algebras:

Theorem 2. *There is a sequence of swaps starting with any simple graph resulting in a minimal extended star.*

This allows us to only need to consider the properties of graph Lie algebras of minimal extended stars. However, it is useful to simplify as many graphs as possible into Dynkin diagrams, so we note another result which can be derived by creating a sequence of swaps:

Theorem 3. *Given an extended star with a leg L with length $l > 4$, its graph Lie algebra is isomorphic to that resulting when L is removed and legs of length 4 and $l - 4$ are added.*



This allows us in many cases to relate minimal extended stars to Dynkin diagrams, or to make extended stars minimal to consider them with other graphs, as seen above.

Decomposing Graph Lie Algebras

Consider a scalar multiple f_α of a monomial e_v such that $f_\alpha^2 = 1$. Then $c_\alpha = (1 + f_\alpha)/2$ is a central idempotent in our graph Lie algebra. A theorem from [3] gives us that for all central monomials e_α in $\mathcal{A}(G)$ and integers $i \in \alpha$,

$$c_\alpha \cdot \mathcal{A}(G), (1 - c_\alpha) \cdot \mathcal{A}(G) \cong \mathcal{A}(G \setminus v_i).$$

We also have that $c_\alpha \cdot \mathcal{A}(G) \oplus (1 - c_\alpha) \cdot \mathcal{A}(G) \cong \mathcal{A}(G)$, since $c(1 - c) = 0$. Therefore, we have that

$$\mathcal{A}(G) \cong \mathcal{A}(G \setminus v_i) \oplus \mathcal{A}(G \setminus v_i).$$

By considering a similar decomposition, while keeping in mind the minimal condition of graph Lie algebras, we determine conditions for which this theorem can be expanded to graph Lie algebras:

Theorem 4. *For a simple graph G and some vertex v_i, we have*

$$\mathfrak{L}(G) \cong \mathfrak{L}(G \setminus v_i) \oplus \mathfrak{L}(G \setminus v_i),$$

provided the following conditions are true:

- *The vector space $e_\alpha \mathfrak{L}(G)$ is a subspace of $\mathfrak{L}(G)$,*
- *The monomial $e_\alpha e_i$ is in $\mathfrak{L}(G \setminus v)$.*

This theorem allows us in some cases to remove a vertex from a graph and consider the graph Lie algebra of the original graph as the direct sum of two copies of the graph Lie algebra of the new graph.

Lemma 2. *If in a graph G, there are two legs of length 1 or two legs of length 3 connected to the same vertex, one of which has a vertex v of degree one, then*

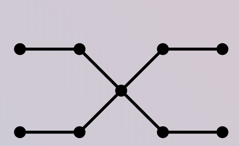
$$\mathcal{L}(G) \cong \mathcal{G} \setminus \sqsubseteq \oplus \mathcal{G} \setminus \sqsubseteq.$$

We can combine this with Theorem 2 to arrive at a stronger result:

Theorem 5. *All graph Lie algebras are equivalent to the graph Lie algebra of a minimal extended star where at most one leg has length 1 and at most one leg has length 3.*

Future Work

The problem of understanding graph Lie algebras of Dynkin diagrams has already been done by Khovanova in [3]. From the results in this paper, in order to describe all graph Lie algebras, we must either continue simplifying the graphs described in Theorem 5 until each can be altered to result in a Dynkin diagram without changing the underlying Lie algebra, or prove this is impossible and describe all new graphs that arise. Although it seems feasible that all graphs can be altered to result in Dynkin diagrams, this cannot be done using our methods. For example, consider the following graph:



The graph Lie algebra of this graph has dimension 255, which can be calculated using a Java program, which is not a power of 2 times the dimension of some Dynkin diagram, so we are currently unable to describe it.

References

[1] J. Goldwasser, X. Wang, and Y. Wu. "Does the lit-only restriction make any difference for the σ-game and σ^+-game?" In: *European J. Combin.* 30.4 (2008), pp. 774–787.

[2] S. Lie. *Theorie Der Transformationsgruppen*. Leipzig: B. G. Teubner Verlag, 1888.

[3] T. G. Khovanova. "Clifford Algebras and Graphs". In: *Geombinatorics* 20.2 (2010), pp. 56–76.