Configuration Spaces and Representation Stability

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Abstract

Given a topology on a set, there is a natural topology on the set of ordered n-tuples of points in that set. This topological space is known as the configuration space, and is the basis of representation stability, which has been studied by Farb and Church in recent years. In particular, permuting the order of our n-tuples yields an automorphism of our configuration space and thus also of its cohomology groups, which provides a representation of the symmetric group. We lay out a process for determining the cohomology groups of configuration spaces of certain orientable manifolds, and propose a description of the generators of the underlying cohomology ring in a way which is conducive to calculating the representations it yields. Finally, we apply these proposals to the case of the complex Grassmannian.

1 Introduction - what is representation stability?

The topic of representation stability, while most explicitly linked to cohomology rings on manifolds, is in fact motivated at least in part by sums over discrete objects. We give an overview of the topic which follows the train of thought of [Far14]. As our motivating example, consider the set of monic square-free degree-n polynomials modulo some prime power q: that is, the set of degree-n polynomials whose irreducible factors are all distinct. Determining the size of this set is relatively simple: we can induct and use the principal of inclusion and exclusion to show that there are $q^{n-1}(q-1)$ such polynomials when $n > 1$. Say however that we wanted to find some weighted sum on these polynomials, based on their factorizations. For example, say that we wanted to find the sum over all of these polynomials of the difference between the number of reducible quadratic factors and the number of irreducible quadratic factors. If we express this weighted sum as a polynomial over q in terms of n, one observation that can be made when considering small values of n is that the degree-$(n-k)$ coefficient is fixed as n varies, assuming that n is sufficiently large relative to k. This is by proxy the "stability" that we wish to express with representation stability. As an example, if we calculate this sum when every polynomial has weight 1 then we are simply counting the size of the set.

Now in order to relate these discrete objects with topological calculations, we note first that a square-free polynomial modulo q has n distinct roots in the algebraic closure of $\mathbb{F}_q$. Furthermore, we see that the Frobenius automorphism $x \mapsto x^q$ fixes this set of roots: in particular, by Fermat's little theorem, taking the variable of these polynomials to the power of q will also take the polynomial itself to the power of q. Considering our discrete set as a set of fixed collections of points in this way allows us to introduce an underlying topological structure. In particular, it may be the case that we can apply a fixed point formula, which may relate both to our discrete set and to some topological structure. Of course, when applying such a formula to our discrete set, we will not be considering the set of potential roots of our polynomials, but rather the set of possible collections of n distinct roots given by a polynomial. In order to describe such a relationship, we must first introduce another object which handles collections of distinct points. The *unordered configuration space* on a space X of n points, denoted by $\mathrm{UConf}_n(X)$, is the topological space of sets of n distinct points on X. Importantly, it was shown that the Grothendieck-Lefschetz fixed point formula can be applied so that in order to count the number of possible sets of n roots of a polynomial in $\mathbb{F}_q$, and thus the number of monic squarefree polynomials of degree n over $\mathbb{F}_q$, we only need to consider the unordered configuration space of n distinct points in $\mathbb{C}$: in particular, this count was shown to equal

$$\sum_{k\leq n}(-1)^k q^{n-k}\dim_{\mathbb{C}}(H^k(\mathrm{UConf}_n(\mathbb{C});\mathbb{C})).$$

That is to say, if the dimensions of the cohomology groups $H^k(\mathrm{UConf}_n(\mathbb{C});\mathbb{C})$ are eventually constant as n increases, then the q^{n-k} coefficients of the polynomial over q which counts the number of square-free polynomials of degree n will eventually be constant as n increases as well. Now unfortunately, the space $\mathrm{UConf}_n(\mathbb{C})$ is a bit unwieldy. However, it turns out that we can process this further by relating it to the *configuration space* $\mathrm{Conf}_n(\mathbb{C})$, which consists of ordered n-tuples $(x_1,\ldots,x_n)$, in contrast with $\mathrm{UConf}_n(\mathbb{C})$ which consists of unordered n-tuples. Note that there is an obvious covering of $\mathrm{UConf}_n(\mathbb{C})$ by $\mathrm{Conf}_n(\mathbb{C})$, and the points of the fiber, which are the permutations of n points in n fixed locations, give a deck transformation of $\mathrm{Conf}_n(\mathbb{C})$, which will thus give us an isomorphism of each $H^k(\mathrm{Conf}_n(\mathbb{C});\mathbb{C})$. Now from [Oss96] we know that these groups are free abelian and finitely generated. That is to say, this transformation will take each generator of such a group to some integer linear combination of generators. This gives us a representation of S_n.

Recall that representations yield class functions via their character. We also have class functions of S_n for any degree-n polynomial, given by the degrees of the irreducible polynomials: in particular, recall that classes of S_n are given by the sizes of the cycles in the elements, so given a polynomial with some set of degrees of irreducible factors, we can assign it a class with cycles with sizes equal to these degrees. In particular then, any polynomial P on the numbers of irreducible factors of various sizes gives us a character of S_n. Our final interesting result then is that summing P over all of our square-free polynomials is in fact equal to a sum of inner products, namely

$$\sum_{k\leq n}(-1)^k q^{n-k}\langle P, H^k(\mathrm{Conf}_n(\mathbb{C});\mathbb{C})\rangle_{S_n}.$$

In particular, we are now interested in finding the complete irreducible decomposition of $H^k(\mathrm{Conf}_n(\mathbb{C});\mathbb{C})$. As a matter of fact, similar processes can be used to calculate weighted sums on other collections of n objects, via various fixed point formulas.

In this paper we will not focus our attention on the existence of similar fixed point formulas in more general cases. Rather, we will propose a process of determining the representations of the configuration spaces of certain orientable manifolds. First we will show that the cohomology groups of the configuration space are closely linked to those of copies of our manifold with finite numbers of points removed, and we will explicitly describe how the cohomology groups of these punctured manifolds relate to those of the original manifold itself. We will then give an explicit relationship between the cohomology ring of the configuration space and that of our punctured manifolds via the tensor product. Once we have set up this relationship, we will be able to propose a set of generators of the cohomology ring of our configuration space in terms of the way in which each point moves along our manifold, which will knock us into place for proposing a calculation of our representation. Finally, we will apply this to the case of the finite-dimensional complex Grassmannian.

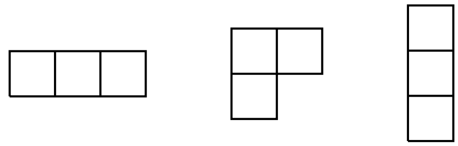

Figure 1: The three Young diagrams of size 3.

2 What is a Young diagram?

To get a better understanding of our representations of S_n, we introduce a set of visually motivated representations, as seen in [Zha08]. A *Young diagram* is a collection of n unit squares inside a rectangular grid such that the only squares with no squares to their left are at the left edge of the grid, and the only squares with no squares above them are at the top of the grid. For example, there are three Young diagrams of size 3, as shown in Figure 1. Given a nonincreasing set $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive values such that $n - \sum \lambda_i \geq \lambda_1$, we are given a Young diagram where the first row has $n - \sum \lambda_i$ elements, and the $(i+1)$-th row has λ_i terms.

From these objects arises a common set of representations of S_n. Say we were to fill up our Young diagram with the values 1 through n, and we were to take note of the set of values in each row. Then a permutation of $\{1, \ldots, n\}$ would give us possibly differing values in each row. Then the partitions of $\{1, \ldots, n\}$ into these rows, ignoring their order within the rows, form the basis of our representation. We see that this is going to be a representation of S_n. For example, if we have the second shape in Figure 1, then if we have 1 and 2 in the top row and 3 in the second row, then permuting 1 and 2 will maintain the corresponding basis element, but permuting 1 and 3 will map this to a different basis element. We then let $V_0(\lambda)$ be the representation given by the Young diagram of λ. For example, in general the permutation representation of S_n is given by $V_0(1)$, and the trivial representation is given by $V_0(0)$. Now recall that in S_n, our polynomials P acted on the numbers of cycles of given sizes in our elements of S_n. Our representations $V_0(\lambda)$ are going to count very similar values. For example, $V_0(\{k\})$ is going to count the number of fixed k-tuples. This is the number of unions of cycles with total sum k. In fact, these representations will allow us to determine a number of other representations. As an example, say we wish to calculate thrice the number of 3-cycles. The only ways to fix 3 elements are via 3 fixed points, one 2-cycle and one fixed point, and one 3-cycle. We see that $V_0(3)$ will count all of these. However, $V_0(2,1)$ will count the first case thrice each and the second case once each. Then $V_0(1,1,1)$ will count the first case 6 times. Thus thrice the number of 3-cycles is given by $3V_0(3) - 3V_0(2,1) + V_0(1,1,1)$. And in general, it is useful for us to describe our representations in terms of our $V_0(\lambda)$. In addition, there is a way to generate an irreducible representation $V(\lambda)$ given a Young diagram, and the representaions given above are going to be linear multiples of these, such that the size of the sum of the terms in our λ is similar. However, for ease of calculation in this paper we will stick with the more easily computable terms above.

3 Homology groups of configuration spaces on certain manifolds

We will consider closed, smooth, connected manifolds M which have finite dimension $d > 2$ and trivial fundamental group. Now since M is a smooth compact manifold, it admits some countable triangulation. We can iteratively remove $(d-1)$-cells, which merges two d-cells into one, so that every d-cell is eventually merged with some given cell after some finite time, since M is connected. This means that we can write M as a CW complex with a single d-cell. Additionally, by Poincaré duality and the universal coefficient theorem we know that $H_{d-1}(M)$ is trivial. For the sake of this paper we will need to take the stronger assumption that M can be written as a CW complex with a single d-cell, which we will call C, and no $(d-1)$-cells. These conditions are not necessarily prohibitive to employing the techniques described below, but having them in place will allow for a smoother explanation. Additionally, we are going to assume that the homology groups of M are known, so that we can express the homology groups of the configuration space in terms of them.

Most useful to us will be the map $\mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M)$ given by taking the first $n-1$ points, which will allow us to calculate our homology groups inductively. To motivate a relationship between the cohomology groups of our configuration space and those of manifolds similar to M, we note that Theorem 5.3 of [Hat] takes the following form here:

Theorem 1 [Hat]. There exists a spectral sequence satisfying the following properties:

- $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ and $E^{r+1}_{p,q} = \ker d_r / \mathrm{im}\, d_r$ at $E^r_{p,q}$,
- the stable terms $E^\infty_{p,k-p}$ are equal to successive quotients of a filtration of $H_k(\mathrm{Conf}_n(M))$,
- $E^2_{p,q} = H_p(\mathrm{Conf}_{n-1}(M); H_q(M_{n-1}))$.

Thus we see it will be useful to calculate the homology groups of M_{n-1}, where M_r is the manifold M with r points removed. We begin by considering those of M_1, which is homeomorphic to $M - C$ by inclusion. Note that by our assumption about the CW structure of M, we know that M_1 has no cells of degree greater than $(d-2)$, so all of the homology groups of M_1 of these degrees are trivial. As for the calculation of the other homology groups, we can apply the following point of Lemma 2.34 of [Hat02]:

Lemma 2 [Hat02]. If X is a CW complex, and X^n is the union of all of the cells of dimension at most n, then the map $H_k(X^n) \to H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$.

In this case we can substitute $M = X$ and $M_1 = X^{d-1}$, so that the homology groups of M_1 of degree at most $d-2$ will be identical to those of M. In particular, we have an isomorphism from the generators of the homology groups of M of degree less than d to the generators of the homology groups of M_1 of the same dimension.

To calculate the homology groups of M_r for $r > 1$, we can employ Mayer-Vietoris sequences. In particular, we can place all of our removed points in C, and then we can huddle them near each other. We can then cover M_r with an open set A which retracts to M_1 and an open set B homeomorphic to to a copy of $\mathbb{R}^d$ with r points removed, where $A \cap B$ is retractible to a copy of S^{d-1}. Intuitively, the set A is the set of everything except for this mass of huddled points, and B is the mass itself. Note that B is retractible to a wedge sum of r copies of S^{d-1}. This is to say that

$$H_k(B) = \begin{cases} \mathbb{Z} & \text{when } k = 0, \\ \mathbb{Z}^r & \text{when } k = d-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now it is clear that M_r can be written as a CW complex with cells of dimension less than d, by adding $(d-1)$-cells to M_1. The structure of this CW complex is unimportant here: we only care that it exists. In particular, its existence implies that $H_d(M_r) = 0$. For $1 < k < d-1$, our Mayer-Vietoris sequence gives an isomorphism $H_k(A) \oplus H_k(B) = H_k(M_r)$, so since in this range A shares homology groups with M and B has trivial homology groups we know that $H_k(M_r) = H_k(M)$. Furthermore, by the structure of our Mayer-Vietoris sequence, our isomorphism between these groups maps corresponding generators to each other. Since $d > 2$, removing points from M will not change the fact that the fundamental group is trivial, so M_r has identical homology groups to M in all dimensions except possibly for $d-1$. In this case, our Mayer-Vietoris sequence gives us the exact

$$H_d(M_r) \to H_{d-1}(A \cap B) \to H_{d-1}(A) \oplus H_{d-1}(B) \to H_{d-1}(M_r) \to H_{d-2}(A \cap B),$$

which simplifies into

$$0 \to \mathbb{Z} \to \mathbb{Z}^n \to H_{d-1}(M_r) \to 0.$$

Note that in particular, the generator of $\mathbb{Z}$ here is the sphere around all of the points which $A \cap B$ retracts to, which in the image is the sum of individual spheres around each point, which form the generators of $\mathbb{Z}^n$. Thus $H_{d-1}(M_r)$ is the quotient $\mathbb{Z}^n/(1, \ldots, 1) = \mathbb{Z}^{n-1}$. Thus we have proved the following:

Theorem 3. If P_0 is the Poincaré polynomial of a manifold M satisfying our conditions, then $P_r = P_0 - x^d + (r-1)x^{d-1}$ is the Poincaré polynomial of M_r. Furthermore, the homology groups of M_r of dimension less than $d-1$ are isomorphic to those of M.

Additionally, we can see the relationship between the generators of M and those of M_r. For M_1, we see that there is a bijection between the generators of the homology and cohomology groups of M_1 and M via inclusion in all dimensions lower than $d-1$. Additionally, by our Mayer-Vietoris sequence there is then a bijection of the generators of the homology and cohomology groups of M_r and M_1 via inclusion in all dimensions below $d-1$. In dimension $d-1$, by Poincaré duality, the universal coefficient theorem and our expression of M as a CW complex with one d-cell, we see that the homology and cohomology groups of M_r are generated by those in B, which are the spheres around various removed points, quotienting out their sum $A \cap B$, so this is generated by the spheres around any point except one.

Now that we are able to explicitly construct these homology groups, giving us the structure of our cohomology groups, we wish to consider the relation between these and our configuration space. We wish to show that the map $\mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M)$ satisfies the conditions of the Leray-Hirsch theorem. In particular, we wish to show that the pullback of this map is a surjection. This is not too difficult: we can simply fix our first $n-1$ points and vary the last point as necessary, which also defines for us a section. We also see from the above that our Betti numbers are all finite. Thus, by applying this theorem to the sequence of maps $\mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M) \to \cdots \to M$, we have that the cohomology ring $H^*(\mathrm{Conf}_n(M))$ is isomorphic to the tensor product $H^*(M) \otimes H^*(M_1) \otimes \cdots \otimes H^*(M_{n-1})$. In particular, the generators of $H^k(\mathrm{Conf}_n(M))$ are the cup products of generators of our $H^{k_j}(M_i)$ with $\sum k_j = k$.

4 Attempting to geometrically understand our cohomology groups

From our formulation of the cohomology ring of our configuration space as a tensor product, we have that the generators of $H^*(\mathrm{Conf}_n(M))$ are given by taking the tensor product of $n-1$ ones with a generator of some $H^*(M_r)$. From the statement of the Leray-Hirsch theorem, if we have some σ^* which is the dual of a generating simplex σ of some $H^*(M_i)$ of dimension k, then when placing it into our configuration space, we are fixing the first i points to give us a term of $H^k(\mathrm{Conf}_{i+1}(M))$, and then taking the pullback of the cohomology groups along the maps $\mathrm{Conf}_n(M) \to \ldots \to \mathrm{Conf}_{i+1}(M)$. In particular, a simplex is mapped to 1 in the pullback when it is homotopic to a simplex on which we fix the first i points, and on which the $(i+1)$-th point varies in the same way as our generator of $H^k(M_i)$.

Note that for a simplex Σ in our configuration space, we are uniquely given a simplex of the same dimension of each individual point x_i by removing the other points, though it may be trivial, and this simplex will be fixed under homotopy. Let this simplex be known as the *restriction* $r_i(\Sigma)$ of Σ to x_i. With this in mind, we can give simplices whose duals we claim are generators of our cohomology ring. Let σ be a k-simplex in M_{i-1}. Fix the first $i-1$ points of our configuration space, and have all of the other points huddle together in a ball, and have this ball vary along σ. Then each of these points vary along σ in M. Thus for any set $\{1,\dots,i-1\}$ of points we wish to fix, and any simplex σ on which we wish x_i to vary, we have a corresponding simplex $\sigma(i)$. This provides a bijection between the generators of the cohomology ring of our configuration space and simplices which these generators are going to map to 1.

From here, we make two conjectures on our structure based on this relationship between the generators of our cohomology ring and our simplices which they map to 1:

Conjecture 4. The duals of simplices $\sigma(i)$ for generators σ of M_{i-1} generate $H^*(\mathrm{Conf}_n(M))$.

Our second conjecture is related to the relationship between our homology groups and our restrictions. To motivate it, we consider the fundamental group of a configuration space on a different manifold, namely a cylinder. Say we have two simplices: one in which two points loop clockwise around the cylinder, and one in which the first of these points loops anticlockwise around the cylinder. In this case it is clear that the sum of these two simplices is homotopic to one in which the second point loops clockwise around our cylinder, and the first remains fixed. In particular, our second claim is as follows:

Conjecture 5. Given a simplex σ of M other than C, consider the simplices $\sigma(i)$ of $\mathrm{Conf}_n(M)$ by moving the $(i+1)$-th through n-th points along σ, which we can formalize by considering σ as a simplex in M_{n-1}, so that our simplices are well-defined. Then a permutation maps these $\sigma(i)$ to sums of other $\sigma(j)$. Additionally, adding two elements of a homology group adds their restrictions.

Now this claim is important when we consider cohomology groups over $\mathbb{C}$: in particular, over $\mathbb{C}$, our groups will be free abelian so our homology and cohomology groups will be isomorphic. Then, the homology group of a sum of some $\sigma(i)$ will be able to be extracted by the restrictions to each point.

Now recall from our analysis of our punctured manifolds M_r that every generator of our configuration space arises either from a generator of M or from a sphere around a point in some M_r. Now recall that not all of the spheres around points need to be considered in order to account for each generator: if we add every sphere around a point for a given M_r we get the sphere which retracts to our set $A \cap B$ from our previous section, and this is our only quotient. Thus we can consider as our generators of this form only the simplices b_{ij} given by the point x_i encompassing a point x_j for $1 < j < i$.

Now even with our conjectures, calculating most representations via these methods is in practice going to be very difficult: in particular, there is a labor-intensity to determining which of our cup products are nonzero. For example, if we have all of the products of the form $\sigma_1(i)\sigma_2(j)$ where σ_1, σ_2 are two generators from M such that $\sigma_1 \smile \sigma_2 = 0$, then we have $\sigma_1(1)\sigma_2(1) = 0$, but $\sigma_1(1)\sigma_2(2)$ ought to not be zero based on our structure of our cohomology ring as a tensor product.

Thus we cannot simply take the tensor product of the representations given by $\sigma_1(i)$ and $\sigma_2(i)$ over varying i. However, given a set of manifolds σ in M with given multiplicities and some set of sphere generators, if we have a cup product of copies of various $\sigma(i)$ in our configuration space which have the same multiplicities of those given, then a permutation is going to give another such product by our conjectures, so we can split our representation via direct sum into these smaller representations. And even with our limitations, these conjectures at least allow us to calculate the representations given by the generators of our cohomology ring, assuming we fully understand our cup product structure: given a generator $\sigma(i)$ from M, our proposal yields that it is always mapped to a sum of $\sigma(j)$ terms, and given a sphere generator, it is clear that it will map to another sphere generator. Thus if we have a product of these σ, and permutation maps each $\sigma(i)$ to a sum of some $\sigma(j)$, then by expanding the tensor product we see that our permutation will be a sum of tensor products of these σ terms with the same multiplicity, assuming this product is nonzero.

Let us first determine the proposed representation given by a generator from M other than C, say σ. Note that by our second conjecture $\sigma(j)-\sigma(j+1)$ is going to have its only nonzero restriction be that of x_j which will be σ, where $\sigma(n+1)=0$. In particular, under our conjectures, given a permutation π, we can write $\sigma(i)$ as $\sum_{j\geq i}[\sigma(i)-\sigma(i+1)]$, so that $\sigma(1)$ maps to itself and for $i>1$ we take $\sigma(i)$ to $\sum_{j\geq i}[\sigma(\pi(i))-\sigma(\pi(i)+1)]$, which we know because these uniquely determine the multiplicity of $\sigma(i)$ in our restrictions. Thus the i,j-th entry of our representation at π would be

$$\begin{cases} 1 & \text{if } \pi^{-1}(i)\geq j \text{ but } \pi^{-1}(i-1)<j, \\ -1 & \text{if } \pi^{-1}(i-1)\geq j \text{ but } \pi^{-1}(i)<j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi^{-1}(0)=0$. In order to better understand this representation, we can calculate its character. In particular, on π our trace will be the number of i such that $\pi^{-1}(i)\geq i$ minus the number of i such that $\pi^{-1}(i)\geq i+1$. This means that our trace will be the number of fixed points of π. In particular, this is isomorphic to the permutation representation $V_0(1)$, which we know is decomposable into one copy of the trivial representation and one copy of the standard representation.

Consider now the set of sphere generators. Note that each sphere generator b_{ij} consists of one point x_i encompassing another point x_j for $1<j<i$. If we swap these two points, we get the same generator: instead of viewing it as one point going around another, view it as the two points going around their midpoint. Now a permutation will map the sphere b_{ij} to $b_{\pi(i)\pi(j)}$. Thus we want a representation whose character counts the number of fixed pairs, which is $V_0(2)$. This is in fact not contingent on our conjectures: we are aware of how these simplices relate without relying on a global structure.

As a side-note, we can also see instances of $V_0(\lambda)$ terms elsewhere in our sphere generators. For example, say we have the cup product $b_{ij}b_{jk}$ for $i>j>k>1$, where b_{ij} is the sphere generator where i encompasses j. This is nonzero as b_{ij} comes from M_{i-1} in our tensor product, whereas b_{jk} comes from M_{j-1}. Say also that d is odd, so that these generators have even dimension and our cup product is commutative. This product is then going to be fixed under a permutation exactly when j is fixed and the pair i,k is fixed. Thus the representation given by this product is going to be $V_0(2,1)$. However, expressing cup products of sphere generators cannot always be immediately be assigned representations $V_0(\lambda)$ in the same way. If we take the cup product $b_{32}b_{43}b_{54}b_{65}$, then clearly 4 needs to be fixed, and the pairs of terms $3,5$ and $2,6$ need to be fixed, but we cannot with this structure fix the order. This is not to say that it cannot be done with some clever summations of course. If we view a nonzero cup like this as a graph on $n-1$ vertices with i,j connected for b_{ij} in our product, then a permutation which fixes this is going to be one which yields a graph isomorphism.

5 Considering the complex Grassmannian

Now we hope to apply Theorem 3 as well as our conjectures to a specific case, namely the complex Grassmannian $Gr(a,b)(\mathbb{C})$ of a-planes in b-space. We will shorten this to Gr for convenience. We will calculate the cohomology of the configuration space and determine as an example the representation that $H^4(\mathrm{Conf}_n(Gr))$ yields assuming our conjectures.

Let us first consider the cohomology groups of Gr itself. These groups were calculated in [Hua14]. We can represent an a-dimensional plane of $\mathbb{C}^b$ as an a-by-b matrix, where the row vectors span our plane. In particular, this matrix can be uniquely expressed in *reduced row echelon forms*, in which the initial nonzero term of each row is 1, the first nonzero dimensions are ordered, and the terms above the leading terms are zero. This allows us to categorize the elements of Gr by the locations of our initial nonzero terms in each row, as seen in Figure 2. These sets are called *Schubert cells*.

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Figure 2: An example of an reduced row echelon form, as seen in [Hua14].

Now each Schubert cell itself is isomorphic to $\mathbb{R}^{2\ell}$, where ℓ is the number of varying terms, since the terms vary in $\mathbb{C}$. This allows us to naturally consider this cell as a 2ℓ-dimensional sphere. This is useful as in Gr, the limit points of these spheres are going to lie on Schubert cells of smaller dimension: by fixing the ratios of the varying terms and having them go to infinity this becomes relatively clear. In particular, this means that we can consider Gr as a CW-complex, with our attaching maps given as just described. It is also useful to notice that there is a single cell C of largest dimension, which is dense in Gr. Since we only have cells of even dimension, the homology groups of Gr are given by the number of cells of a given dimension. In particular, the group $H_k(Gr;\mathbb{C})$ is trivial when k is odd and otherwise is equal to $\mathbb{C}^s$ where s is the number of sets of nondecreasing integers $0 \le x_1, \ldots, x_a \le b-a$ whose sum is $k/2$. Thus by our work in Section 3, we have calculated the cohomology groups of the corresponding punctured manifolds as well.

From this, assuming our conjectures, we have the information necessary to propose a structure for the representation given by $H^4(\mathrm{Conf}_n((Gr))$. We assume here that $a, b-a > 1$. We ignore the calculation of the representation of H^2 as it is too small to be worth noticing: we only have a single generator σ_2 of $H^2(Gr)$, so H^2 would give us the permutation representation $V_0(1)$, as we already showed. For H^4, we must consider cup products as well. We see in Proposition 2.4 of [Hua14] that the cup square of σ_2 is nonzero given our assumption on $b-a$. There are two ways to sum to 2 with a nondecreasing sequence, namely 2 and $1+1$. Thus $H^4(Gr)$ is two-dimensional, so it is generated by σ_2^2 and some σ_4, and these pairs give us the generators of $H^4(\mathrm{Conf}_n(Gr), \mathbb{C})$ as well. Of course in the latter case we do not consider these strictly as generators in Gr, but rather as representative of the cup products of various corresponding terms in the configuration space. Now in our configuration space, we note that our sphere generators have dimension at least 7, so we need not worry about them here. Now by our conjectures any generator $\sigma_4(i)$ maps to a sum of $\sigma_4(j)$ under a permutation and any generator

$\sigma_2(i)\sigma_2(j)$ is going to map to a sum of products $\sigma_2(k)\sigma_2(m)$ under our permutation, so we can split our representation via direct sum into our representations given by these two sets. Now we have already analyzed representations of sets of generators such as those given by σ_4, which will be isomorphic to the permutation representation $V_0(1)$. This leaves us to consider the images under permutations of our generators $\sigma_2(i)\sigma_2(j)$, which we do by expressing the image of each multiplicand as a sum of terms like it.

Let $\{m_{ij}\}$ be the matrix given by the permutation π. First, our trace element for $\sigma_2^2(i)$ will be m_{ii}^2. Then our trace element for $\sigma_2(i)\sigma_2(j)$ where $i < j$ will be the sum of two things: first it will contain the product of the trace elements of $\sigma_2(i)$ and $\sigma_2(j)$, or $m_{ii}m_{jj}$, but it will also contain the multiplicity of $\sigma_2(j)$ in $\sigma_2(\pi(i))$ times the multiplicity of $\sigma_2(i)$ in $\sigma_2(\pi(j))$, that is, $m_{ij}m_{ji}$. Now say we double our representation. Then for every pair $i \neq j$ we have a copy of $m_{ij}m_{ji}$, and for every $i \neq j$ we have a copy of $m_{ii}m_{jj}$. On top of this, we have two copies of m_{ii}^2 for each i. Adding one copy to our first set gives us the trace of π^2, and adding one copy to the second gives us the square of the trace of π. Now recall that these traces count fixed points. Thus our trace of π^2 will be the number of fixed points of π plus twice the number of 2-cycles in π, while the trace of π will be the number of fixed points of π. Thus dividing back by two gives us the number of 2-cycles plus the number of fixed points plus the number of fixed points choose 2, or $V_0(2) + V_0(1)$. Thus along with our representations of our σ_4 terms, assuming our conjectures the representation of $H^4(\mathrm{Conf}_n(Gr), \mathbb{C})$ is given by $V_0(2) + 2V_0(1)$.

6 Future work

Of course, the most obvious next step for this work is to prove our two conjectures, or if they are incorrect attempt to modify them so that our representation calculations can still be performed. This would allow us for a large number of manifolds to understand how exactly the representation is going to be calculated and what it will be equal to.

In terms of big-picture improvements, the next most useful advancement would be to see whether we can use a modified form of the Leray-Hirsch theorem to calculate the cohomology ring of our configuration spaces over a more general manifold. This would hopefully give us a similarly convenient method for viewing the cohomology ring of the configuration space in terms of those of the manifold, as was done here.

Secondly, while it is useful to have these representations on hand, we have yet to relate them explicitly to weighted sums, as was done on polynomials in a finite field in our example. Work on this is relevant to the étale cohomology and is central to our relationship between objects over finite fields and similar objects over the complex numbers. In fact, the choice of the complex Grassmannian as an example was partially motivated by its potential to have similar fixed point formulas applied to it, possibly relating it to the Grassmannian over a finite field.

Finally, it may be useful for us to consider representations of cup products more closely. As a matter of fact, it seems to be within reach to give an explicit description for the representation given by the cohomology group of the configuration space of a manifold M satisfying our conditions, assuming we are given the structure of the cohomology ring of M.

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