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RESEARCH ARTICLE

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Koszul cohomology and support of local cohomology modules of complete intersections

Michael Gintz | **Wenliang Zhang**

Department of Mathematics, Statistics, and Computer Science, Chicago, Illinois, USA

Correspondence
Wenliang Zhang, Department of Mathematics, Statistics, and Computer Science, Chicago, IL 60607, USA.
Email: wlzhang@uic.edu

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Abstract

Let R be a Noetherian commutative ring and $f_1, \dots, f_c$ be a regular sequence in R. We introduce a framework to study $\mathrm{Supp}(H^j_I(R/(f_1, \dots, f_c)))$ by linking the Koszul cohomology of $H^j_I(R)$ on the sequence $f_1, \dots, f_c$ and local cohomology modules $H^j_I(R/(f_1, \dots, f_c))$. As an application, we prove that if R is a Noetherian regular ring of prime characteristic p and $f_1, f_2 \in R$ form a regular sequence, then $\mathrm{Supp}(H^j_I(R/(f_1, f_2)))$ is Zariski-closed for each integer j and each ideal I.

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1 | INTRODUCTION

Let R be a Noetherian commutative ring and I be an ideal. Let Γ_I denote the I-torsion functor defined via:

$$\Gamma_I(M) = \{z \in M \mid I^t z = 0 \text{ for some integer } t\}; \quad \Gamma_I(M \xrightarrow{f} N) = \Gamma_I(M) \xrightarrow{f|_{\Gamma_I(M)}} \Gamma_I(N).$$

It turns out that Γ_I is left-exact; the jth local cohomology of an R-module M, denoted by $H^j_I(M)$, is defined as $\mathbb{R}^j\Gamma_I(M)$; that is,

$$H^j_I(M) \cong H^j(0 \to Q^\bullet),$$

where $0 \to M \to Q^\bullet$ is an injective resolution of M. It can be calculated by a Čech complex; cf. Section 2 for details.

Since the theory of local cohomology was introduced in [3], the study of finiteness properties of these modules, as well as their vanishing, has become an active research topic. The interested reader is referred to [6] for a list of inspiring open questions on vanishing and finiteness properties of local cohomology modules. One of these question asks whether the set of associated primes of $H^j_I(R)$ is finite for each integer j and each ideal I in R. Some positive answers are known: When R is a regular ring of equi-characteristic p [8], when R is either a regular local ring of equi-characteristic 0 or a regular affine ring of equi-characteristic 0 [13], when R is an unramified regular local ring of mixed characteristic [15], when R is a smooth $\mathbb{Z}$-algebra [1], and when either $\dim(R)$ or j is sufficiently small (cf. [2, 4, 11]). Examples in [9, 16, 17] show that local cohomology modules may have infinitely many associated primes. However, the following question (cf. [7, p. 3194]) remains open:

Question 1.1. Let R be a Noetherian commutative ring and I be an ideal. Is $\mathrm{Supp}(H^j_I(R))$ Zariski-closed in $\mathrm{Spec}(R)$ for each integer j?

Note that $\mathrm{Supp}(H^j_I(R))$ being Zariski-closed is equivalent to having finitely many *minimal* associated primes. Hence Question 1.1 concerns with a finiteness property of local cohomology modules. Ref. [7, p. 3195] states that "Clearly, this question is of central importance in the study of cohomological dimension and understanding the local–global properties of local cohomology." Some positive answers to Question 1.1 are known: when $j = 2$ and $H^t_I(R) = 0$ for all $t > 2$ [7, Theorem 1.2] and when $R = S/(f)$ where S is a Noetherian regular ring of prime characteristic p [5, 10].

One of the main results of this article is the following:

Theorem 1.2 (= Theorem 6.5). *Let S be a Noetherian regular ring of prime characteristic p and f_1, f_2 be a regular sequence in S. Set $R = S/(f_1, f_2)$. Then* $\mathrm{Supp}(H^j_I(R))$ *is Zariski-closed for each integer j and each ideal I.*

Our strategy to prove Theorem 1.2 is to link the Koszul cohomology groups of $H^j_I(R)$ on a sequence $\underline{f}$ to the local cohomology modules $H^i_I(R/(\underline{f}))$ via a double complex. To wit, let R be a Noetherian ring and $\underline{f} = f_1, \dots, f_c$ be a sequence of elements. Let $I = (g_1, \dots, g_t)$ be an ideal in R. Let $\check{C}^\bullet(\underline{g}; N)$ denote the Čech complex of an R-module N on the sequence $\underline{g}$ and let $K^\bullet(\underline{f}; N)$ denote the Koszul (co)complex of an R-module N on the sequence $\underline{f}$. Let $\mathbf{D}$ denote the double complex whose ith row is the Čech complex $\check{C}^\bullet(\underline{g}; K^i(\underline{f}; R))$ and whose jth column is the Koszul (co)complex $K^\bullet(\underline{f}; C^j(\underline{g}; R))$. Then there is a spectral sequence

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H^j_I(R))) \Rightarrow H^{i+j}(T^\bullet)$$

associated with $\mathbf{D}$, where $T^\bullet$ denotes the total complex of $\mathbf{D}$ (cf. Section 2 for details). The following theorem provides a framework to study $\mathrm{Supp}(H^k_I(R/(\underline{f})))$ via investigating $H^i(K^\bullet(\underline{f}; H^j_I(R)))$.

Theorem 1.3 (= Theorem 2.4). *Let R be a Noetherian ring, $I = (g_1, \dots, g_t)$ be an ideal, and $f_1, \dots, f_c$ be a sequence of elements in R. Let $E_2^{\bullet,\bullet}$ be as above. Assume that*

(1) $\mathrm{Supp}(E_\infty^{i,j})$ *are Zariski-closed for all integers i, j, and that*
(2) *$f_1, \dots, f_c$ form a regular sequence in R.*

Then $\mathrm{Supp}(H^k_I(R/(f_1, \dots, f_c)))$ *is Zariski-closed for each integer k.*

This article is organized as follows. In Section 2, we introduce and study a double complex which links the Koszul cohomology of $H^j_I(R)$ on a sequence $\underline{f}$ and the local cohomology modules $H^j_I(R/(\underline{f}))$ and prove Theorem 1.3; Section 2 is characteristic-free and does not require R to be regular. In Section 3, we introduce the notion of the (Frobenius) truncation of Čech complexes which is one of the main technical tools in this article. In Sections 4 and 5, we prove that $H^i(K^\bullet(f_1,f_2;\mathcal{M}))$ has Zariski-closed support when f_1,f_2 form a regular sequence in regular ring R of prime characteristic p and $\mathcal{M}$ is an F-finite F-module. In Section 6, we complete the proof of Theorem 1.2.

2 | A KOSZUL–ČECH DOUBLE COMPLEX AND RELATED SPECTRAL SEQUENCES

Let R be a commutative Noetherian ring and $f_1,\dots,f_c$ and $g_1,\dots,g_t$ be two sequences of elements in R. Set $I=(g_1,\dots,g_t)$ to be the ideal generated by $g_1,\dots,g_t$. For each R-module N,

(1) we denote by $K^\bullet(\underline{f};N)$ the Koszul co-complex of N on the elements $f_1,\dots,f_c$, which is the R-dual of the Koszul complex $K_\bullet(\underline{f};N)$, and
(2) we denote by $\check{C}^\bullet(\underline{g};N)$ the Čech complex of N on $g_1,\dots,g_t$:

$$0\to N\xrightarrow{\delta^0}\oplus_{i=1}^t N_{g_i}\xrightarrow{\delta^1}\oplus_{i_1<i_2}N_{g_{i_1}g_{i_2}}\xrightarrow{\delta^2}\cdots\to N_{g_1\cdots g_t}\to 0,$$

where δ^i defined via $\delta^i: N_{g_{j_1}\cdots g_{j_i}}\to N_{g_{\ell_1}\cdots g_{\ell_{i+1}}}$ is defined as

$$\delta^i\left(\frac{z}{g^n_{j_1}\cdots g^n_{j_i}}\right)=\begin{cases}(-1)^{s-1}\frac{z}{g^n_{j_1}\cdots g^n_{j_i}} & \text{when } j_1\cdots j_i=\ell_1\cdots\hat{\ell}_s\cdots\ell_{i+1}\\ 0 & \text{otherwise}\end{cases}$$

for each $z\in N$. Note that $H^j(\check{C}^\bullet(\underline{g};N))\cong H^j_I(N)$.

Definition 2.1. The double complex, denoted by $\mathbf{D}:=D(K^\bullet(\underline{f});\check{C}^\bullet(\underline{g}))$, is the double complex whose ith row is the Čech complex $\check{C}^\bullet(\underline{g};K^i(\underline{f};R))$ and whose jth column is the Koszul (co)complex $K^\bullet(\underline{f};C^j(\underline{g};R))$.

We will denote the total complex of $\mathbf{D}$ by $T^\bullet$.

Example 2.2 (When $t=2$). The most relevant case for this article is when $t=2$ and we would like to spell out the double complex as follows. The Koszul (co)complex $K^\bullet(f_1,f_2;N)$ is the following for each R-module N:

$$0\to N\xrightarrow{\begin{pmatrix}f_1\\ f_2\end{pmatrix}}N^{\oplus 2}\xrightarrow{\begin{pmatrix}-f_2 & f_1\end{pmatrix}}N\to 0$$

The Koszul–Čech double complex in this case is the following:

$$
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & R & \longrightarrow & \bigoplus_j R_{g_j} & \longrightarrow & \bigoplus_{j_1<j_2} R_{g_{j_1}g_{j_2}} & \longrightarrow & \cdots & \longrightarrow & R_{g_1\cdots g_t} & \longleftrightarrow & 0 \\
 & & \uparrow{\scriptstyle (-f_2\ \ f_1)} & & \uparrow{\scriptstyle (-f_2\ \ f_1)} & & \uparrow{\scriptstyle (-f_2\ \ f_1)} & & & & \uparrow{\scriptstyle (-f_2\ \ f_1)} & & \\
0 & \longrightarrow & R^{\oplus 2} & \longrightarrow & (\bigoplus_j R_{g_j})^{\oplus 2} & \longrightarrow & (\bigoplus_{j_1<j_2} R_{g_{j_1}g_{j_2}})^{\oplus 2} & \longrightarrow & \cdots & \longrightarrow & (R_{g_1\cdots g_t})^{\oplus 2} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \binom{f_1}{f_2}} & & \uparrow{\scriptstyle \binom{f_1}{f_2}} & & \uparrow{\scriptstyle \binom{f_1}{f_2}} & & & & \uparrow{\scriptstyle \binom{f_1}{f_2}} & & \\
0 & \longrightarrow & R & \longrightarrow & \bigoplus_j R_{g_j} & \longrightarrow & \bigoplus_{j_1<j_2} R_{g_{j_1}g_{j_2}} & \longrightarrow & \cdots & \longrightarrow & R_{g_1\cdots g_t} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array}
\tag{2.0.1}
$$

Remark 2.3. As discussed in [18, section 5.1], there are two spectral sequences associated with our complex $D(K^\bullet(\underline{f}); \check{C}^\bullet(\underline{g}))$.

One of them comes from taking horizontal differentials (in the Čech complexes) first and then vertical differentials (in the resulting Koszul co-complexes). The resulting spectral sequence is:

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H_I^j(R)) \Rightarrow H^{i+j}(T^\bullet).$$

Recall that $T^\bullet$ is the total complex of $D(K^\bullet(\underline{f}); \check{C}^\bullet(\underline{g}))$.

The other one comes from doing differentials the other way around (considering vertical differentials and then horizontal differentials):

$$'E_2^{i,j} := H_I^i(H^j(K^\bullet(\underline{f}; R))) \Rightarrow H^{i+j}(T^\bullet).$$

The following theorem, one of our main technical tools, indicates the connection between $\mathrm{Supp}(E_\infty^{i,j})$ and $\mathrm{Supp}(H_I^k(R/(f_1, \dots, f_s)))$ when $f_1, \dots, f_c$ form a regular sequence in R.

Theorem 2.4. *Assume that*

(1) $\mathrm{Supp}(E_\infty^{i,j})$ *are Zariski-closed for all integers* i, j, *and that*
(2) $f_1, \dots, f_c$ *form a regular sequence in* R.

Then $\mathrm{Supp}(H_I^k(R/(f_1, \dots, f_c)))$ *is Zariski-closed for each integer* k.

Proof. The convergence

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H_I^j(R))) \Rightarrow H^{i+j}(T^\bullet)$$

amounts to a filtration of $H^k(T^\bullet)$ for each k:

$$0 \subseteq F^k H^k(T^\bullet) \subseteq F^{k-1} H^k(T^\bullet) \subseteq \cdots \subseteq F^1 H^k(T^\bullet) \subseteq F^0 H^k(T^\bullet) = H^k(T^\bullet)$$

such that $F^i H^k(T^\bullet)/F^{i+1} H^k(T^\bullet) \cong E_\infty^{i,n-i}$ (with $F^k H^k(T^\bullet) \cong E_\infty^{k,0}$).

Since $E_\infty^{i,j}$ is Zariski-closed for all integers i, j, the Zariski-closedness of $\mathrm{Supp}(H^k(T^\bullet))$ follows from the filtration of $H^k(T^\bullet)$.

The assumption that $f_1, \ldots, f_s$ form a regular sequence in R implies that ${}'E_2^{\bullet,\bullet}$ has only one nonzero row in which the entries are $H^i_I(R/(f_1, \ldots, f_c))$. Consequently, this spectral sequence collapses, providing for each k the isomorphism

$$H^k_I(R/(f_1, \ldots, f_c)) \cong H^k(T^\bullet)$$

which shows that $\mathrm{Supp}(H^k_I(R/(f_1, \ldots, f_c))$ is Zariski-closed. □

In Section 6, we will prove that $\mathrm{Supp}(E_\infty^{i,j})$ are Zariski-closed for all integers i, j when R is *regular* of prime characteristic p and $E_\infty^{i,j}$ are associated with the double complex (2.0.1). One of our technical tools is to truncate the Čech complex.

3 | TRUNCATED ČECH COMPLEXES

In this section we introduce (Frobenius) truncated Čech complexes, one of the main technical tools needed in this article.

Let R be a Noetherian commutative ring of prime characteristic $p > 0$ and let $g \in R$ be an element in R. We will use $R \cdot \frac{1}{g^{p^e}}$ to denote the cyclic R-submodule of R_f generated by $\frac{1}{g^{p^e}}$, and we will call $R \cdot \frac{1}{g^{p^e}}$ the eth (Frobenius) truncation of R_g. (Our convention is to consider $R \cdot \frac{1}{g}$ as the 0th Frobenius truncation of R_g.)

Note that $R \cdot \frac{1}{g^{p^e}}$ is a finitely generated R-module; this finiteness plays a crucial role in this article.

Remark 3.1. Let $g_1, \ldots, g_t$ be elements in R. Recall that $\check{C}^\bullet(\underline{g}; R)$, the Čech complex of R on $g_1, \ldots, g_t$, is constructed as follows:

$$0 \to R \to \bigoplus_{j=1}^{t} R_{g_j} \to \cdots \to \bigoplus_{j_1 < \cdots < j_i} R_{g_{j_1} \cdots g_{j_i}} \xrightarrow{\delta^i} \bigoplus_{j_1 < \cdots < j_{i+1}} R_{g_{j_1} \cdots g_{j_{i+1}}} \to \cdots \to R_{g_1 \cdots g_t} \to 0,$$

where δ^i defined via $\delta^i : R_{g_{j_1} \cdots g_{j_i}} \to R_{g_{\ell_1} \cdots g_{\ell_{i+1}}}$ is defined as

$$\delta^i\left(\frac{r}{g_{j_1}^n \cdots g_{j_i}^n}\right) = \begin{cases} (-1)^{s-1} \frac{r}{g_{j_1}^n \cdots g_{j_i}^n} & \text{when } j_1 \cdots j_i = \ell_1 \cdots \hat{\ell}_s \cdots \ell_{i+1} \\ 0 & \text{otherwise.} \end{cases} \tag{3.0.1}$$

Then it is clear that the image of the restriction of δ^i on $R \cdot \frac{1}{g_{j_1}^{p^e} \cdots g_{j_i}^{p^e}}$ is contained in $R \cdot \frac{1}{g_{\ell_1}^{p^e} \cdots g_{\ell_{i+1}}^{p^e}}$. Consequently, if one replaces each module in the Cech complex $C^\bullet(\underline{g}; R)$ by its eth truncation, then one will get a complex

$$0 \to R \to \bigoplus_{j=1}^{t} R \cdot \frac{1}{g_j^{p^e}} \to \bigoplus_{j_1 < j_2} R \cdot \frac{1}{g_{j_1}^{p^e} g_{j_2}^{p^e}} \to \cdots \tag{3.0.2}$$

Definition 3.2. The complex (3.0.2) is called the eth truncation of the Čech complex $\check{C}^\bullet(\underline{g};R)$ and will be denoted by $\check{C}^\bullet(\underline{g};R)_e$ or $\check{C}^\bullet_e$ when the elements $g_1,\dots,g_t$ are clear from the context. The ith term in $\check{C}^\bullet(\underline{g};R)_e$ will be denoted by $\check{C}^i(\underline{g};R)_e$ and the ith differential in $\check{C}^\bullet(\underline{g};R)_e$ will be denoted by δ^i_e.

For each element $\eta \in \ker(\delta^i)$ (respectively, $\eta \in \ker(\delta^i_e)$), its image in $H^i(\check{C}^\bullet(\underline{g};R))$ (respectively, $H^i(\check{C}^\bullet(\underline{g};R)_e)$) will be denoted by $[\eta]$.

Let R be a Noetherian ring of prime characteristic p. Let $R^{(e)}$ be the additive group of R regarded as an R-bimodule with the usual left R-action and with the right R-action defined by $r'r = r^{p^e}r'$ for all $r \in R$ and $r' \in R^{(e)}$. The eth Peskine–Szipro functor $\mathbf{F}^e$ is defined via

$$\mathbf{F}^e(M) = R^{(e)} \otimes_R M \qquad \mathbf{F}^e(M \xrightarrow{\phi} N) = R^{(e)} \otimes_R M \xrightarrow{\mathbf{1}\otimes\phi} R^{(e)} \otimes_R N.$$

When $e = 1$, we will denote $\mathbf{F}^1$ by $\mathbf{F}$.

Note that, when R is regular, $R^{(e)}$ is a faithfully flat R-module and hence $\mathbf{F}^e$ is an exact functor for each $e \geqslant 1$ ([12]).

Proposition 3.3. *Let R be a Noetherian regular ring of prime characteristic $p > 0$ and let $\mathbf{F}$ denote the Peskine–Szpiro functor. Then*

(1) $\mathbf{F}(R \cdot \frac{1}{g}) \cong R \cdot \frac{1}{g^p}$ *for every $g \in R$.*
(2) $\mathbf{F}(\check{C}^\bullet(\underline{g};R)_e) \cong \check{C}^\bullet(\underline{g};R)_{e+1}$ *for all sequences of elements $\underline{g} = g_1,\dots,g_t$.*

Proof. Note that $\mathbf{F}$ is an exact functor since R is regular.

To prove the first part, it suffices to note that the R linear map

$$\theta : \mathbf{F}(R\cdot\frac{1}{g}) = R^{(1)} \otimes_R R\cdot\frac{1}{g} \xrightarrow{r'\otimes\frac{r}{g}\mapsto\frac{r'r^p}{g^p}} R\cdot\frac{1}{g^p}$$

admits an inverse

$$R\cdot\frac{1}{g^p} \xrightarrow{\frac{r}{g^p}\mapsto r\otimes\frac{1}{g}} R^{(1)} \otimes_R R\cdot\frac{1}{g} = \mathbf{F}\left(R\cdot\frac{1}{g}\right).$$

The second part follows from the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{F}(R\cdot\frac{1}{g_{j_1}^{p^e}\cdots g_{j_i}^{p^e}}) & \longrightarrow & \mathbf{F}(R\cdot\frac{1}{g_{\ell_1}^{p^e}\cdots g_{\ell_{i+1}}^{p^e}}) \\
\downarrow & & \downarrow \\
R\cdot\frac{1}{g_{j_1}^{p^{e+1}}\cdots g_{j_i}^{p^{e+1}}} & \longrightarrow & R\cdot\frac{1}{g_{\ell_1}^{p^{e+1}}\cdots g_{\ell_{i+1}}^{p^{e+1}}}
\end{array}$$

where the horizontal maps are induced by the ith differential (3.0.1) in the Čech complex and the vertical maps are the isomorphisms in the first part applied to the cases when $g = g_{j_1}^{p^e}\cdots g_{j_i}^{p^e}$ and when $g = g_{\ell_1}^{p^e}\cdots g_{\ell_{i+1}}^{p^e}$, respectively. □

For the rest of this article, we will denote by θ the isomorphisms

$$\mathbf{F}^e(\check{C}^j(\underline{g};R)) \xrightarrow{\sim} \check{C}^j(\underline{g};R), \quad \mathbf{F}(\check{C}^j(\underline{g};R)_e) \xrightarrow{\sim} \check{C}^j(\underline{g};R)_{e+1} \quad \text{and} \quad \mathbf{F}^e(\check{C}^j(\underline{g};R)_0) \xrightarrow{\sim} \check{C}^j(\underline{g};R)_e.$$

The natural inclusion $R \cdot \frac{1}{g_{j_1}^{p^e} \cdots g_{j_i}^{p^e}} \to R \cdot \frac{1}{g_{j_1}^{p^{e+1}} \cdots g_{j_i}^{p^{e+1}}}$ induces a chain map between the truncated Čech complexes: $\check{C}^\bullet(\underline{g};R)_e \to \check{C}^\bullet(\underline{g};R)_{e+1}$ and hence induces an R-module homomorphism $H^i(\check{C}^\bullet(\underline{g};R)_e) \to H^i(\check{C}^\bullet(\underline{g};R)_{e+1})$. This produces a directed system:

$$H^i(\check{C}^\bullet(\underline{g};R)_0) \to H^i(\check{C}^\bullet(\underline{g};R)_1) \to \cdots \to H^i(\check{C}^\bullet(\underline{g};R)_e) \to \cdots$$

whose direct limit is isomorphic to $H^i_I(R)$.

Each element in $H^i_I(R)$ can be represented by a cohomological class of the form $[\dots, \frac{r}{g_{j_1}^n \cdots g_{j_i}^n}, \dots]$. Let $H^j_I(R)_e$ be the R-submodule of $H^j_I(R)$ generated by classes $[\dots, \frac{r}{g_{j_1}^n \cdots g_{j_i}^n}, \dots]$ with $n \leqslant p^e$. Then $H^j_I(R)_e$ is precisely the image of $H^i(\check{C}^\bullet(\underline{g};R)_e)$ in $H^j_I(R)$; consequently $H^i_I(R)_e$ is finitely generated. Furthermore, one can check that

$$H^i_I(R)_e \cong \frac{\ker(\delta^i_e)}{\operatorname{image}(\delta^{i-1}) \cap \ker(\delta^i_e)} \quad \text{and} \quad \mathbf{F}(H^i_I(R)_e) \cong H^i_I(R)_{e+1}. \tag{3.0.3}$$

For the rest of this article, whenever it is clear from the context, we will write $\check{C}^\bullet(\underline{g})$, or even $\check{C}^\bullet$, instead of $\check{C}^\bullet(\underline{g};R)$.

One can replace the Čech complex with its (Frobenius) truncations in Definition 2.1 to form the double complex

$$\mathbf{D}_e := D(K^\bullet(\underline{f^{p^e}}); \check{C}^\bullet(\underline{g})_e)$$

for each integer $e \geqslant 0$:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & R & \to & \bigoplus_j R \cdot \frac{1}{g_j^{p^e}} & \to & \bigoplus_{j_1<j_2} R \cdot \frac{1}{g_{j_1}^{p^e} g_{j_2}^{p^e}} & \to & \cdots & \to & R \cdot \frac{1}{g_1^{p^e} \cdots g_s^{p^e}} & \to & 0 \\
 & & \uparrow {\scriptstyle (-f_2^{p^e}\ f_1^{p^e})} & & \uparrow {\scriptstyle (-f_2^{p^e}\ f_1^{p^e})} & & \uparrow {\scriptstyle (-f_2^{p^e}\ f_1^{p^e})} & & & & \uparrow {\scriptstyle (-f_2^{p^e}\ f_1^{p^e})} & & \\
0 & \to & R^{\oplus 2} & \to & (\bigoplus_j R \cdot \frac{1}{g_j^{p^e}})^{\oplus 2} & \to & (\bigoplus_{j_1<j_2} R \cdot \frac{1}{g_{j_1}^{p^e} g_{j_2}^{p^e}})^{\oplus 2} & \to & \cdots & \to & (R \cdot \frac{1}{g_1^{p^e} \cdots g_s^{p^e}})^{\oplus 2} & \to & 0 \\
 & & \uparrow {\scriptstyle \binom{f_1^{p^e}}{f_2^{p^e}}} & & \uparrow {\scriptstyle \binom{f_1^{p^e}}{f_2^{p^e}}} & & \uparrow {\scriptstyle \binom{f_1^{p^e}}{f_2^{p^e}}} & & & & \uparrow {\scriptstyle \binom{f_1^{p^e}}{f_2^{p^e}}} & & \\
0 & \to & R & \to & \bigoplus_j R \cdot \frac{1}{g_j^{p^e}} & \to & \bigoplus_{j_1<j_2} R \cdot \frac{1}{g_{j_1}^{p^e} g_{j_2}^{p^e}} & \to & \cdots & \to & R \cdot \frac{1}{g_1^{p^e} \cdots g_s^{p^e}} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array} \tag{3.0.4}$$

A priori, one can form the double complex $D(K^\bullet(\underline{f^{p^e}}); \check{C}^\bullet(\underline{g})_{e'})$ for two different integers e and e'. Since this is not needed in this article, we opt not to explore it here.

We will denote the total complex of (3.0.4) by $T_e^\bullet$. When taking the horizontal differentials (those in the truncated Čech complexes) and then the vertical differentials in (3.0.4), one obtains a spectral sequence:

$$E_{2,e}^{i,j} := H^i(K^\bullet(\underline{f}; H^j(\check{C}_e)) \Rightarrow H^{i+j}(T_e^\bullet). \tag{3.0.5}$$

We will denote the differentials in (3.0.5) by

$$\varphi_{2,e}^{i,j} : E_{2,e}^{i,j} \to E_{2,e}^{i+2,j-1}.$$

Since $\mathbf{F}$ is an exact functor, one can check $\mathbf{F}^e(K^\bullet(\underline{f}; R)) \cong K^\bullet(\underline{f^{p^p}}; R)$ for any sequence $\underline{f}$ of elements in R. On the other hand, according to Proposition 3.3, $\mathbf{F}^e(\check{C}^\bullet(\underline{g})_0) \cong \check{C}^\bullet(\underline{g})_e$ for any sequence $\underline{g}$ of elements in R. Consequently, the double complex $\mathbf{D}_e$ can be obtained by applying $\mathbf{F}^e$ to $\mathbf{D}_0$.

According to Theorem 2.4, it suffices to analyze the double complex $\mathbf{D}$. One of our motivations to introduce the double complexes $\mathbf{D}_e$ is that a great deal of information of $\mathbf{D}$ is already encoded in $\mathbf{D}_0$ in which every module is finitely generated. As shown in the sequel, one can link $\mathbf{D}_0$ with $\mathbf{D}$ using the Peskine–Szpiro functor $\mathbf{F}$. This link is rather intricate since $\mathbf{D}_0$ is directly linked with $\mathbf{D}_e$ via $\mathbf{F}^e$ (the differentials in the Koszul (co)complex in $\mathbf{D}_e$ come from the elements $f_1^{p^e}, f_2^{p^e}$, not f_1, f_2).

4 | KOSZUL COHOMOLOGY OF F-FINITE F-MODULES

Let R be a Noetherian *regular* ring of prime characteristic $p > 0$. In this section, we will investigate $E_2^{i,j}$ in the $E_2^{\bullet,\bullet}$-page coming from the double complex $\mathbf{D}$ has Zariski-closed support; that is the Koszul cohomology $H^i(K^\bullet(\underline{f}; H_I^j(R)))$. Instead of local cohomology modules $H_I^j(R)$, we will consider all F-finite F-modules. To this end, we begin by recalling the definition and basic facts of F-modules (cf. [14]).

(1) An R-module $\mathscr{M}$ is an *F-module* if there is an R-module isomorphism

$$\theta : \mathscr{M} \to \mathbf{F}(\mathscr{M}) = R^{(1)} \otimes_R \mathscr{M}$$

called the structure isomorphism.

(2) If $(\mathscr{M}, \theta_{\mathscr{M}})$ and $(\mathscr{N}, \theta_{\mathscr{N}})$ are F-modules, then an *F-module morphism* from $(\mathscr{M}, \theta_{\mathscr{M}})$ to $(\mathscr{N}, \theta_{\mathscr{N}})$ consists of the following commutative diagram:

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\varphi} & \mathscr{N} \\ \downarrow{\scriptstyle \theta_{\mathscr{M}}} & & \downarrow{\scriptstyle \theta_{\mathscr{N}}} \\ R^{(1)} \otimes_R \mathscr{M} & \xrightarrow{\mathbf{1}\otimes\varphi} & R^{(1)} \otimes_R \mathscr{N} \end{array}$$

We will simply write this F-module morphism as $\varphi : \mathscr{M} \to \mathscr{N}$ whenever the context is clear.

(3) A *generating morphism* of an F-module is an R-module homomorphism $\beta : M \to \mathbf{F}(M)$, where M is an R-module, such that $\mathcal{M}$ is the direct limit of the top row of the following commutative diagram,

$$\begin{array}{ccccccc} M & \xrightarrow{\beta} & \mathbf{F}(M) & \xrightarrow{\mathbf{F}(\beta)} & \mathbf{F}^2(M) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathbf{F}(\beta)} & & \downarrow & & \\ \mathbf{F}(M) & \xrightarrow{\mathbf{F}(\beta)} & \mathbf{F}^2(M) & \xrightarrow{\mathbf{F}^2(\beta)} & \mathbf{F}^3(M) & \longrightarrow & \cdots \end{array}$$

and the structure isomorphism $\theta : \mathcal{M} \to \mathbf{F}(\mathcal{M})$ is induced by the vertical morphism in the diagram.

(4) An F-module $\mathcal{M}$ is *F-finite* if it admits a generating morphism $\beta : M \to \mathbf{F}(M)$ where M is a finitely generated R-module.

(5) Each F-finite F-module $\mathcal{M}$ admits an injective generating morphism $\beta : M \hookrightarrow \mathbf{F}(M)$ where M is a finitely generated R-module; (M, β) is called a root of $\mathcal{M}$.

(6) For each $f \in R$, the localization R_f is an F-finite F-module.

(7) Given elements $g_1, \dots, g_s \in R$, the Čech complex $\check{C}^\bullet(\underline{g}; R)$ is a complex in the category of F-finite F-modules; that is, each module $\check{C}^j$ is an F-finite F-module and the differentials δ^j in this complex are F-module morphisms.

(8) $\ker(\delta^j)$ and $\operatorname{image}(\delta^j)$ are F-finite F-modules and consequently $H^j_I(R)$ is an F-finite F-module for each integer j and each ideal I in R.

Let $\mathcal{M}$ be an F-finite F-module and $\beta : M \hookrightarrow \mathbf{F}(M)$ is a root. Let $R^b \xrightarrow{A} R^a \to M \to 0$ be a presentation of M where A is an $a \times b$ matrix whose entries are elements of R. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} R^b & \xrightarrow{A} & R^a & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle \beta} & & \\ R^b & \xrightarrow{A^{[p]}} & R^a & \longrightarrow & F(M) & \longrightarrow & 0 \end{array}$$

where $A^{[p]}$ denotes the matrix whose entries are the pth powers of the corresponding entries in A and U is an $a \times a$ matrix with entries in R. To ease notation, we will denote this diagram by

$$\operatorname{coker}(A) \xrightarrow{U} \operatorname{coker}(A^{[p]}).$$

Let $f_1, \dots, f_c$ be a sequence of elements in R and let $H^i(\underline{f}; -)$ denote the ith Koszul cohomology functor. That is,

$$H^c(\underline{f}; N) \cong N/(\underline{f}N) \quad \text{and} \quad H^0(\underline{f}; N) \cong \bigcap_{j=1}^{t} \ker(N \xrightarrow{f_j} N)$$

for each R-module N.

Theorem 4.1. *For each F-finite F-module $\mathscr{M}$, we have that* $\mathrm{Supp}(H^c(\underline{f};\mathscr{M}))$ *and* $\mathrm{Supp}(H^0(\underline{f};\mathscr{M}))$ *are Zariski-closed, where $\underline{f} = \{f_1, \dots, f_c\}$ is an arbitrary sequence of elements in R.*

Before we proceed to the proof, we remark that the special case of Theorem 4.1 when $c = 1$ and $\mathscr{M} = H^j_I(R)$ recovers [5, Theorem 1.1] and [10, Theorem 7.1(c)].

Proof of Theorem 4.1. To treat the 0th Koszul cohomology, we consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{coker}(A) & \xrightarrow{U} & \mathrm{coker}(A^{[p]}) & \xrightarrow{U^{[p]}} & \mathrm{coker}(A^{[p^2]}) & \xrightarrow{U^{[p^2]}} & \cdots \\
\left\downarrow \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} \right. & & \left\downarrow \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} \right. & & \left\downarrow \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} \right. & & \\
\mathrm{coker}(A)^{\oplus c} & \xrightarrow{U^{\oplus c}} & \mathrm{coker}(A^{[p]})^{\oplus c} & \xrightarrow{(U^{[p]})^{\oplus c}} & \mathrm{coker}(A^{[p^2]})^{\oplus c} & \xrightarrow{(U^{[p^2]})^{\oplus c}} & \cdots
\end{array} \tag{4.0.1}$$

Each square in this commutative diagram

$$\begin{array}{ccc}
\mathrm{coker}(A^{[p^e]}) & \xrightarrow{U^{[p^e]}} & \mathrm{coker}(A^{[p^{e+1}]}) \\
\left\downarrow \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} \right. & & \left\downarrow \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} \right. \\
\mathrm{coker}(A^{[p^e]})^{\oplus c} & \xrightarrow{U^{[p^e]}} & \mathrm{coker}(A^{[p^{e+1}]})^{\oplus c}
\end{array}$$

commutes since $U^{[p^e]} f_j = f_j U^{[p^e]}$ for each f_j. Therefore (4.0.1) is a commutative diagram. One can check that the direct limit of (4.0.1) is

$$\mathscr{M} \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix}} \mathscr{M}^{\oplus c}.$$

It follows from the proof of [10, Theorem 7.1] that

$$\mathrm{Supp}(\ker(\mathscr{M} \xrightarrow{f_j} \mathscr{M})) = \mathrm{Supp}\left(\frac{(\ker(U^{[p^j]} \cdots U)) :_{R^a} f_j}{\ker(U^{[p^j]} \cdots U)} \right), \ j \gg 0.$$

Consequently

$$\mathrm{Supp}(H^0(\underline{f};\mathscr{M})) = \mathrm{Supp}\left(\frac{(\ker(U^{[p^j]} \cdots U)) :_{R^a} (f_1, \dots, f_c)}{\ker(U^{[p^j]} \cdots U)} \right), \ j \gg 0$$

which is Zariski-closed.

To handle the cth Koszul cohomology, we consider the following diagram:

$$\begin{array}{ccccccc}
\operatorname{coker}(A)^{\oplus c} & \xrightarrow{U^{\oplus c}} & \operatorname{coker}(A^{[p]})^{\oplus c} & \xrightarrow{(U^{[p]})^{\oplus c}} & \operatorname{coker}(A^{[p^2]})^{\oplus c} & \xrightarrow{(U^{[p^2]})^{\oplus c}} & \cdots \\
\Big\downarrow{\scriptstyle (f_1,\ldots,f_c)} & & \Big\downarrow{\scriptstyle (f_1,\ldots,f_c)} & & \Big\downarrow{\scriptstyle (f_1,\ldots,f_c)} & & \\
\operatorname{coker}(A) & \xrightarrow{U} & \operatorname{coker}(A^{[p]}) & \xrightarrow{U^{[p]}} & \operatorname{coker}(A^{[p^2]}) & \xrightarrow{U^{[p^2]}} & \cdot
\end{array} \tag{4.0.2}$$

Each square in this commutative diagram

$$\begin{array}{ccc}
\operatorname{coker}(A^{[p^e]})^{\oplus c} & \xrightarrow{(U^{[p^e]})^{\oplus c}} & \operatorname{coker}(A^{[p^{e+1}]})^{\oplus c} \\
\Big\downarrow{\scriptstyle (f_1,\ldots,f_c)} & & \Big\downarrow{\scriptstyle (f_1,\ldots,f_c)} \\
\operatorname{coker}(A^{[p^e]}) & \xrightarrow{U^{[p^e]}} & \operatorname{coker}(A^{[p^{e+1}]})
\end{array}$$

commutes since $U^{[p^e]}f_j = f_j U^{[p^e]}$ for each f_j. Therefore (4.0.2) is a commutative diagram. One can check that the direct limit of (4.0.2) is

$$\mathcal{M}^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} \mathcal{M}.$$

Each element in $\mathcal{M}$ can be represented by an element $z \in \operatorname{coker}(A^{[p^e]})$ for some e. Let $\mathfrak{p}$ be a prime ideal of R. This element becomes 0 in $H^c(\underline{f};\mathcal{M})_{\mathfrak{p}}$ if and only if there is an integer j such that

$$(U^{[p^{e+j}]} \cdots U^{[p^e]})z \in \Big(\operatorname{image}((f_1,\ldots,f_c)) + \operatorname{image}(A^{[p^{e+j+1}]})\Big).$$

Therefore,

$$(H^c(\underline{f};\mathcal{M})_{\mathfrak{p}} = 0 \Leftrightarrow \bigcup_j \Big((\operatorname{image}((f_1,\ldots,f_c)) + \operatorname{image}(A^{[p^{e+j+1}]})) :_{R^a} (U^{[p^{e+j}]} \cdots U^{[p^e]})\Big)_{\mathfrak{p}} = R^a_{\mathfrak{p}},\ \forall\, e.$$

Recall that M is assumed to have a presentation $R^b \xrightarrow{A} R^a \to M \to 0$.

Since

$$\begin{aligned}
&\Big((\operatorname{image}((f_1,\ldots,f_c)) + \operatorname{image}(A^{[p^{e+j+1}]})) :_{R^a} (U^{[p^{e+j}]} \cdots U^{[p^e]})\Big)^{[p]} \\
&= (\operatorname{image}((f_1^p,\ldots,f_c^p)) + \operatorname{image}(A^{[p^{e+j+2}]})) :_{R^a} (U^{[p^{e+j+1}]} \cdots U^{[p^{e+1}]}) \\
&\subseteq (\operatorname{image}((f_1,\ldots,f_c)) + \operatorname{image}(A^{[p^{e+j+2}]})) :_{R^a} (U^{[p^{e+j+1}]} \cdots U^{[p^{e+1}]}),
\end{aligned}$$

one can check that

$$(H^t(\underline{f};\mathcal{M})_{\mathfrak{p}} = 0 \Leftrightarrow \bigcup_j \Big((\operatorname{image}((f_1,\ldots,f_c)) + \operatorname{image}(A^{[p^{e+j+1}]})) :_{R^a} (U^{[p^{e+j}]} \cdots U^{[p^e]})\Big)_{\mathfrak{p}} = R^a_{\mathfrak{p}}$$

if and only if

$$(H^t(\underline{f};\mathscr{M})_{\mathfrak{p}} = 0 \Leftrightarrow \bigcup_j \Big((\text{image}((f_1,\dots,f_c)) + \text{image}(A^{[p^{j+1}]})) :_{R^a} (U^{[p^j]}\cdots U)\Big)_{\mathfrak{p}}$$

$$= R^a_{\mathfrak{p}} \text{ (that is when } e = 0).$$

This proves that

$$\operatorname{Supp}(H^t(\underline{f};\mathscr{M})) = \operatorname{Supp}\left(\frac{R^a}{(\text{image}((f_1,\dots,f_c)) + \text{image}(A^{[p^{j+1}]})) :_{R^a} (U^{[p^j]}\cdots U)}\right)$$

which is clearly Zariski-closed. □

The most relevant case to this article is when $\underline{f}$ is a regular sequence in R. We pose the following question:

Question 4.2. Let R be a Noetherian regular ring of primes characteristic p and $\underline{f}$ be a regular sequence in R. Is it true that $\operatorname{Supp}(H^i(K^\bullet(\underline{f};\mathscr{M})))$ is Zariski-closed for each integer i and each F-finite F-module $\mathscr{M}$?

To the best our knowledge, Question 4.2 is open as stated. In the next section, we will show that it has an affirmative answer when $\underline{f} = f_1, f_2$.

5 | REGULAR SEQUENCES OF LENGTH 2

In this section, we consider the case when $t = 2$; that is, when R is an F-finite Noetherian regular ring of prime characteristic, f_1, f_2 form a regular sequence in R and $\mathscr{M}$ is an F-finite F-module. The main goal in this section is to prove the following result:

Theorem 5.1. *$\operatorname{Supp}(H^1(K^\bullet(f_1, f_2;\mathscr{M})))$ is Zariski-closed for every F-finite F-module $\mathscr{M}$ and arbitrary elements f_1, f_2 in R.*

Before we can prove Theorem 5.1, we would like to consider a special case of it:

Theorem 5.2. *Assume that an F-finite F-module $\mathscr{M}$ is (f_1, f_2)-torsion. Then $\operatorname{Supp}(H^1(K^\bullet(f_1, f_2;\mathscr{M})))$ is Zariski-closed.*

Proof. It follows from the following long exact sequence of Koszul cohomology

$$0 \leftarrow H^2(K^\bullet(f_1,f_2;\mathscr{M})) \leftarrow H^1(K^\bullet(f_1;\mathscr{M})) \xleftarrow{f_2} H^1(K^\bullet(f_1;\mathscr{M}))$$

$$\leftarrow H^1(K^\bullet(f_1,f_2;\mathscr{M})) \leftarrow H^0(K^\bullet(f_1;\mathscr{M})) \xleftarrow{f_2} H^0(K^\bullet(f_1;\mathscr{M})) \leftarrow H^0(K^\bullet(f_1,f_2;\mathscr{M})) \leftarrow 0$$

that

$$\begin{aligned}&\operatorname{Supp}(H^1(K^\bullet(f_1,f_2;\mathcal{M})))\\ &=\operatorname{Supp}(\operatorname{coker}(H^0(K^\bullet(f_1;\mathcal{M}))\xrightarrow{f_2}H^0(K^\bullet(f_1;\mathcal{M}))))\bigcup\operatorname{Supp}(\ker(H^1(K^\bullet(f_1;\mathcal{M}))\xrightarrow{f_2}H^1(K^\bullet(f_1;\mathcal{M})))).\end{aligned}$$

Note that swapping f_1 and f_2 does not affect $H^1(K^\bullet(f_1,f_2;\mathcal{M}))$; consequently

$$\operatorname{Supp}(\operatorname{coker}(H^0(K^\bullet(f_2;\mathcal{M}))\xrightarrow{f_1}H^0(K^\bullet(f_2;\mathcal{M}))))\subseteq\operatorname{Supp}(H^1(K^\bullet(f_1,f_2;\mathcal{M}))).$$

Hence

$$\begin{aligned}\operatorname{Supp}(H^1(K^\bullet(f_1,f_2;\mathcal{M})))=&\operatorname{Supp}(\operatorname{coker}(H^0(K^\bullet(f_1;\mathcal{M}))\xrightarrow{f_2}H^0(K^\bullet(f_1;\mathcal{M}))))\\ &\bigcup\operatorname{Supp}(\operatorname{coker}(H^0(K^\bullet(f_2;\mathcal{M}))\xrightarrow{f_1}H^0(K^\bullet(f_2;\mathcal{M}))))\\ &\bigcup\operatorname{Supp}(\ker(H^1(K^\bullet(f_1;\mathcal{M}))\xrightarrow{f_2}H^1(K^\bullet(f_1;\mathcal{M})))).\end{aligned}$$

First we treat $\operatorname{Supp}(\ker(H^1(K^\bullet(f_1;\mathcal{M}))\xrightarrow{f_2}H^1(K^\bullet(f_1;\mathcal{M}))))$. Note that

$$\ker(H^1(K^\bullet(f_1;\mathcal{M}))\xrightarrow{f_2}H^1(K^\bullet(f_1;\mathcal{M})))\cong\ker\left(\frac{\mathcal{M}}{f_1\mathcal{M}}\xrightarrow{f_2}\frac{\mathcal{M}}{f_1\mathcal{M}}\right)\cong\frac{f_1\mathcal{M}:_{\mathcal{M}}f_2}{f_1\mathcal{M}}.$$

Let L denote a root of $\mathcal{M}$; that is, L is finitely generated R-submodule of $\mathcal{M}$ equipped with an injective R-module morphism $\beta : L\to\mathbf{F}(L)$ that generates the F-module $\mathcal{M}$. We will set $L_e := \mathbf{F}^e(L)\subseteq\mathcal{M}$ and view L_e as a submodule of L_{e+1} via the injective R-module morphism $F^e(\beta)$. Note that $\mathcal{M}=\cup_{e\geqslant 1}L_e$.

Claim 1. $\operatorname{Supp}\left(\frac{f_1\mathcal{M}:_{\mathcal{M}}f_2}{f_1\mathcal{M}}\right)=\bigcup_{e\geqslant 1}\operatorname{Supp}\left(\frac{f_1\mathcal{M}\cap L_e:_{L_e}f_2}{f_1\mathcal{M}\cap L_e}\right)$.

Assume that $\frac{f_1\mathcal{M}:_{\mathcal{M}}f_2}{f_1\mathcal{M}}=0$. For each $e\geqslant 1$ and $z_e\in(f_1\mathcal{M}\cap L_e:_{L_e}f_2)$, it follows that $f_2z_e\in f_1\mathcal{M}\cap L_e\subseteq f_1\mathcal{M}$ and consequently $z_e\in f_1\mathcal{M}\cap L_e$. This shows that $\frac{f_1\mathcal{M}\cap L_e:_{L_e}f_2}{f_1\mathcal{M}\cap L_e}=0$ for each e; that is,

$$\operatorname{Supp}\left(\frac{f_1\mathcal{M}:_{\mathcal{M}}f_2}{f_1\mathcal{M}}\right)\supseteq\bigcup_{e\geqslant 1}\operatorname{Supp}\left(\frac{f_1\mathcal{M}\cap L_e:_{L_e}f_2}{f_1\mathcal{M}\cap L_e}\right).$$

On the other hand, assume that $\frac{f_1\mathcal{M}\cap L_e:_{L_e}f_2}{f_1\mathcal{M}\cap L_e}=0$ for each e. For each $z\in f_1\mathcal{M}:_{\mathcal{M}}f_2\subseteq\mathcal{M}$, there is an e such that $z\in L_e$. Consequently $f_2z\in f_1\mathcal{M}\cap L_e$ and hence $z\in f_1\mathcal{M}\cap L_e\subseteq f_1\mathcal{M}$ by the assumption. This shows that $\frac{f_1\mathcal{M}:_{\mathcal{M}}f_2}{f_1\mathcal{M}}=0$; that is,

$$\operatorname{Supp}\left(\frac{f_1\mathcal{M}:_{\mathcal{M}}f_2}{f_1\mathcal{M}}\right)\subseteq\bigcup_{e\geqslant 1}\operatorname{Supp}\left(\frac{f_1\mathcal{M}\cap L_e:_{L_e}f_2}{f_1\mathcal{M}\cap L_e}\right).$$

This finishes the proof of our Claim 1.

Claim 2. $\operatorname{Supp}(\frac{f_1\mathscr{M}\cap L:_L f_2}{f_1\mathscr{M}\cap L}) = \bigcup_{e\geqslant 1}\operatorname{Supp}\left(\frac{f_1\mathscr{M}\cap L_e:_{L_e} f_2}{f_1\mathscr{M}\cap L_e}\right)$.

It suffices to show that if $\frac{f_1\mathscr{M}\cap L:_L f_2}{f_1\mathscr{M}\cap L} = 0$, then $\frac{f_1\mathscr{M}\cap L_e:_{L_e} f_2}{f_1\mathscr{M}\cap L_e} = 0$ for each $e \geqslant 1$. Applying the functor $\mathbf{F}^e(-)$ to the assumption $\frac{f_1\mathscr{M}\cap L:_L f_2}{f_1\mathscr{M}\cap L} = 0$, one deduces that $\frac{f_1^{p^e}\mathscr{M}\cap L_e:_{L_e} f_2^{p^e}}{f_1^{p^e}\mathscr{M}\cap L_e} = 0$; that is,

$$f_1^{p^e}\mathscr{M}\cap L_e :_{L_e} f_2^{p^e} = f_1^{p^e}\mathscr{M}\cap L_e.$$

Let z_e be an element in $f_1\mathscr{M}\cap L_e :_{L_e} f_2$. Since $\mathscr{M}$ is (f_1, f_2)-torsion, there exists an integer j such that $f_2^{jp^e} z_e = 0$. Since $f_2^{p^e}(f_2^{(j-1)p^e} z_e) = 0 \in f_1^{p^e}\mathscr{M}\cap L_e$, it follows that $f_2^{(j-1)p^e} z_e \in f_1^{p^e}\mathscr{M}\cap L_e$. Repeating this process, one deduces that $z_e \in f_1^{p^e}\mathscr{M}\cap L_e \subseteq f_1\mathscr{M}\cap L_e$. This proves our Claim 2.

Combining these two claims shows that

$$\operatorname{Supp}(\ker(H^1(K^\bullet(f_1;\mathscr{M})) \xrightarrow{f_2} H^1(K^\bullet(f_1;\mathscr{M})))) = \operatorname{Supp}(\frac{f_1\mathscr{M}\cap L :_L f_2}{f_1\mathscr{M}\cap L})$$

which is Zariski-closed as L is finitely generated.

It remains to prove that

$$\operatorname{Supp}(\operatorname{coker}(H^0(K^\bullet(f_1;\mathscr{M})) \xrightarrow{f_2} H^0(K^\bullet(f_1;\mathscr{M})))) \bigcup \operatorname{Supp}(\operatorname{coker}(H^0(K^\bullet(f_2;\mathscr{M})) \xrightarrow{f_1} H^0(K^\bullet(f_2;\mathscr{M}))))$$

is Zariski-closed (which will complete the proof of our lemma).

Note that

$$H^0(K^\bullet(f_1;\mathscr{M})) \cong (0:_{\mathscr{M}} f_1) \quad \text{and} \quad H^0(K^\bullet(f_2;\mathscr{M})) = (0:_{\mathscr{M}} f_2)$$

and consequently

$$\operatorname{coker}(H^0(K^\bullet(f_1;\mathscr{M})) \xrightarrow{f_2} H^0(K^\bullet(f_1;\mathscr{M}))) \cong \frac{(0:_{\mathscr{M}} f_1)}{f_2(0:_{\mathscr{M}} f_1)},$$
$$\operatorname{coker}(H^0(K^\bullet(f_2;\mathscr{M})) \xrightarrow{f_1} H^0(K^\bullet(f_2;\mathscr{M}))) \cong \frac{(0:_{\mathscr{M}} f_2)}{f_1(0:_{\mathscr{M}} f_2)}.$$

Since $\mathscr{M} = \cup_{e\geqslant 0} L_e$, it is straightforward to check that

$$\operatorname{Supp}\left(\frac{(0:_{\mathscr{M}} f_1)}{f_2(0:_{\mathscr{M}} f_1)}\right) = \bigcup_e \operatorname{Supp}\left(\frac{(0:_{L_e} f_1)}{f_2(0:_{\mathscr{M}} f_1)\cap(0:_{L_e} f_1)}\right),$$
$$\operatorname{Supp}\left(\frac{(0:_{\mathscr{M}} f_2)}{f_1(0:_{\mathscr{M}} f_2)}\right) = \bigcup_e \operatorname{Supp}\left(\frac{(0:_{L_e} f_2)}{f_1(0:_{\mathscr{M}} f_2)\cap(0:_{L_e} f_2)}\right). \tag{5.0.1}$$

Since L is finitely generated and is (f_1, f_2)-torsion, there is an integer e_0 such that

(1) $f_1^{p^{e_0}} L = f_2^{p^{e_0}} L = 0$,
(2) $f_1(0:_{\mathscr{M}} f_2)\cap(0:_L f_2) = f_1(0:_{L_{e_0}} f_2)\cap(0:_L f_2)$, and

(3) $f_2(0:_{\mathscr{M}} f_1)\cap(0:_L f_1) = f_2(0:_{L_{e_0}} f_1)\cap(0:_L f_1)$.

Note that $f_1^{p^{e_0}}L = f_2^{p^{e_0}}L = 0$ implies that

$$f_1^{p^{e_0+e}}L_e = f_2^{p^{e_0+e}}L_e = 0 \tag{5.0.2}$$

for each integer $e \geqslant 1$.

Claim 3.

$$\begin{aligned}
&\operatorname{Supp}\left(\frac{(0:_{\mathscr{M}} f_1)}{f_2(0:_{\mathscr{M}} f_1)}\right)\cup\operatorname{Supp}\left(\frac{(0:_{\mathscr{M}} f_2)}{f_1(0:_{\mathscr{M}} f_2)}\right)\\
&=\operatorname{Supp}\left(\frac{(0:_L f_1)}{f_2(0:_{\mathscr{M}} f_1)\cap(0:_L f_1)}\right)\cup\operatorname{Supp}\left(\frac{(0:_{L_{e_0}} f_1)}{f_2(0:_{\mathscr{M}} f_1)\cap(0:_{L_{e_0}} f_1)}\right)\\
&\cup\operatorname{Supp}\left(\frac{(0:_L f_2)}{f_1(0:_{\mathscr{M}} f_2)\cap(0:_L f_2)}\right)\cup\operatorname{Supp}\left(\frac{(0:_{L_{e_0}} f_2)}{f_1(0:_{\mathscr{M}} f_2)\cap(0:_{L_{e_0}} f_2)}\right).
\end{aligned}$$

The inclusion $\supseteq$ follows from (5.0.1); it remains to show $\subseteq$. To this end, assume that

- $(0:_L f_1)\subseteq f_2(0:_{\mathscr{M}} f_1)$,
- $(0:_{L_{e_0}} f_1)\subseteq f_2(0:_{\mathscr{M}} f_1)$,
- $(0:_L f_2)\subseteq f_1(0:_{\mathscr{M}} f_2)$, and
- $(0:_{L_{e_0}} f_2)\subseteq f_1(0:_{\mathscr{M}} f_2)$,

and we need to show $(0:_{\mathscr{M}} f_1) = f_2(0:_{\mathscr{M}} f_1)$ and $(0:_{\mathscr{M}} f_2) = f_1(0:_{\mathscr{M}} f_2)$.

Note it follows from our choice of e_0 that $(0:_L f_1)\subseteq f_2(0:_{L_{e_0}} f_1)$ and $(0:_L f_2)\subseteq f_1(0:_{L_{e_0}} f_2)$.

Given the symmetry between f_1 and f_2, it suffices to show that $(0:_{\mathscr{M}} f_1) = f_2(0:_{\mathscr{M}} f_1)$.

Let $z\in(0:_{\mathscr{M}} f_1)$ be an arbitrary nonzero element. Then $z\in(0:_{L_e} f_1)$ for an integer e since $\mathscr{M} = \cup_e L_e$. It follows from (5.0.2) that $f_2^{p^{e_0+e}}z = 0$ since $f_2^{p^{e_0+e}}L_e = 0$. That is,

$$z\in(0:_{L_e} f_2^{p^{e_0+e}})\subseteq(0:_{L_{e_0+e}} f_2^{p^{e_0+e}}) = \mathbf{F}^{e_0+e}(0:_L f_2)\subseteq\mathbf{F}^{e_0+e}(f_1(0:_{L_{e_0}} f_2)) = f_1^{p^{e_0+e}}(0:_{L_{2e_0+e}} f_2^{e_0+e})).$$

Hence, there is an $y\in(0:_{L_{2e_0+e}} f_2^{e_0+e}))$ such that $z = f_1^{p^{e_0+e}}y = f_1^{p^{e_0+e}-1}(f_1y)$. Note that

$$f_1^{p^{e_0+e}}(f_1y) = f_1f_1^{p^{e_0+e}}y = f_1z = 0$$

which implies that

$$f_1y\in(0:_{L_{2e_0+e}} f_1^{p^{e_0+e}}) = \mathbf{F}^{e_0+e}((0:_{L_{e_0}} f_1))\subseteq\mathbf{F}^{e_0+e}(f_2(0:_{\mathscr{M}} f_1)) = f_2^{p^{e_0+e}}(0:_{\mathscr{M}} f_1^{p^{e_0+e}}).$$

Thus, there is a $w\in(0:_{\mathscr{M}} f_1^{p^{e_0+e}})$ such that $f_1y = f_2^{p^{e_0+e}}w$. Set

$$x = f_1^{p^{e_0+e}-1}f_2^{p^{e_0+e}-1}w.$$

Then

$$f_2 x = f_2 f_2^{p^{e_0+e}-1} f_1^{p^{e_0+e}-1} x = f_1^{p^{e_0+e}-1} f_2^{p^{e_0+e}} w = f_1^{p^{e_0+e}-1} f_1 y = f_1^{p^{e_0+e}} y = z$$

and

$$f_1 x = f_1 f_2^{p^{e_0+e}-1} f_1^{p^{e_0+e}-1} w = f_2^{p^{e_0+e}-1} f_1^{p^{e_0+e}} w = 0$$

since $f_1^{p^{e_0+e}} w = 0$ by the choice of w. This proves that $z = f_2 x$ and $x \in (0 :_{\mathscr{M}} f_1)$; that is, $z \in f_2(0 :_{\mathscr{M}} f_1)$ and hence completes the proof of our Claim 3.

Note that Claim 3 implies $\mathrm{Supp}(\frac{(0:_{\mathscr{M}} f_1)}{f_2(0:_{\mathscr{M}} f_1)}) \cup \mathrm{Supp}(\frac{(0:_{\mathscr{M}} f_2)}{f_1(0:_{\mathscr{M}} f_2)})$ is Zariski-closed since both L and L_{e_0} are finitely generated.

Combining our 3 claims completes the proof of our theorem. □

We now return to the general case when $\mathscr{M}$ is an arbitrary F-finite F-module. Let Γ denote $\Gamma_{(f_1,f_2)}(\mathscr{M})$. The short exact sequence

$$0 \to \Gamma \to \mathscr{M} \to \mathscr{M}/\Gamma \to 0$$

induces an exact sequence on Koszul cohomology

$$0 = H^0(K^\bullet(\underline{f}; \mathscr{M}/\Gamma(\mathscr{M}))) \to H^1(K^\bullet(\underline{f}; \Gamma)) \to H^1(K^\bullet(\underline{f}; \mathscr{M})) \to H^1(K^\bullet(\underline{f}; \mathscr{M}/\Gamma)) \xrightarrow{\delta} H^2(K^\bullet(\underline{f}; \Gamma)). \tag{5.0.3}$$

The connecting morphism δ can be constructed as follows. Each element in $H^1(\underline{f}; \mathscr{M}/\Gamma)$ can be represented by a pair (a, b) with $-f_2 a + f_1 b = 0 \in \mathscr{M}/\Gamma$ and $a, b \in \mathscr{M}/\Gamma$; equivalently, each element in $H^1(\underline{f}; \mathscr{M}/\Gamma)$ can be represented by a pair (a, b) in $\mathscr{M} \oplus \mathscr{M}$ such that $-f_2 a + f_1 b \in \Gamma$. Then

$$\delta(a, b) = \overline{-f_2 a + f_1 b} \in \frac{\Gamma}{(f_1, f_2)\Gamma} \cong H^2(K^\bullet(\underline{f}; \Gamma)).$$

Following notation in the proof of Lemma 5.2, we denote by L a root of $\mathscr{M}$; that is, L is a finitely generated R-module with an injective R-module morphism $\beta : L \to \mathbf{F}(L)$ that generates $\mathscr{M}$.

Lemma 5.3. $\mathrm{Supp}(\ker(\delta)) = \mathrm{Supp}\left(\frac{(f_1\mathscr{M} \cap L :_L f_2)}{(f_1\mathscr{M} \cap L :_L f_2) \cap (\cup_{j \geqslant 0}((f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)))}\right)$. *In particular, it is Zariski-closed.*

Proof. First we would like to prove that following claim.

Claim. $\mathrm{Supp}(\ker(\delta)) = \mathrm{Supp}\left(\frac{(f_1\mathscr{M} :_{\mathscr{M}} f_2)}{(f_1\mathscr{M} :_{\mathscr{M}} f_2) \cap (\cup_j (f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j))}\right)$.

To prove our claim, we show that

$$\ker(\delta) = 0 \Leftrightarrow (f_1\mathscr{M} :_{\mathscr{M}} f_2) = (f_1\mathscr{M} :_{\mathscr{M}} f_2) \cap (\cup_j (f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j)).$$

Each element in $\ker(\delta)$ can be represented by (a, b) with $a, b \in \mathscr{M}$ such that $f_1 b - f_2 a \in (f_1, f_2)\Gamma$. That is, there are $u, v \in \Gamma$ such that $f_2 b - f_1 a = f_1 u + f_2 v$. By replacing a, b with $a + u, b - v$ (which does not change the images of a, b in $\mathscr{M}/\Gamma$), one can assume that $f_2 a = f_1 b$.

Assume that $\ker(\delta) = 0$. Given each $a \in (f_1\mathscr{M} :_{\mathscr{M}} f_2)$, there is an element $b \in \mathscr{M}$ such that $f_2 a = f_1 b$ and hence (a, b) produces an element in $\ker(\delta)$ which is zero by our assumption. Hence there is an element $c \in \mathscr{M}$ such that

$$(f_1 c, f_2 c) = (a, b) \in (\mathscr{M}/\Gamma)^{\oplus 2};$$

that is, there is an integer j such that $f_1^j(f_1 c - a) = 0$ which implies that $a \in (f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j)$. This proves that $(f_1\mathscr{M} :_{\mathscr{M}} f_2) = (f_1\mathscr{M} :_{\mathscr{M}} f_2) \cap (\cup_j (f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j))$.

On the other hand, assume that $(f_1\mathscr{M} :_{\mathscr{M}} f_2) = (f_1\mathscr{M} :_{\mathscr{M}} f_2) \cap (\cup_j (f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j))$. Let (a, b) be an element in $\ker(\delta)$. According to the discussion above, we can assume that $f_2 a = f_1 b$ and hence $a \in (f_1\mathscr{M} :_{\mathscr{M}} f_2)$. It follows from the assumption that there is an integer j such that $f_1^j a = f_1^{j+1} c$. Then

$$f_1^{j+1}(f_2 c - b) = f_2 f_1^{j+1} a - f_1^{j+1} b = f_2 f_1^j a - f_1^{j+1} b = f_1^{j+1} b - f_1^{j+1} b = 0$$

and hence

$$(f_1 c, f_2 c) = (a, b) \in (\mathscr{M}/\Gamma)^{\oplus 2}$$

which shows that $(a, b) = 0 \in H^1(\underline{f}; \mathscr{M}/\Gamma)$. This finishes the proof of our claim. □

It remains to show that

$$\operatorname{Supp}\left(\frac{(f_1\mathscr{M} :_{\mathscr{M}} f_2)}{(f_1\mathscr{M} :_{\mathscr{M}} f_2) \cap (\cup_j (f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j))}\right)$$
$$= \operatorname{Supp}\left(\frac{(f_1\mathscr{M} \cap L :_L f_2)}{(f_1\mathscr{M} \cap L :_L f_2) \cap (\cup_{j \geqslant 0}((f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)))}\right)$$

which is equivalent to proving

$$(f_1\mathscr{M} :_{\mathscr{M}} f_2) \subseteq \cup_{j \geqslant 0} (f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j) \Leftrightarrow (f_1\mathscr{M} \cap L :_L f_2) \subseteq \cup_{j \geqslant 0} (f_1^{j+1}\mathscr{M} \cap L :_L f_1^j).$$

We begin with the implication $\Rightarrow$. Assume that $(f_1\mathscr{M} :_{\mathscr{M}} f_2) \subseteq \cup_{j \geqslant 0} (f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j)$. Let $a \in (f_1\mathscr{M} \cap L :_L f_2)$ be an arbitrary element. Then, as $L \subseteq \mathscr{M}$, there is an integer j and element $c \in \mathscr{M}$ such that $f_1^j a = f_1^{j+1} c$. This shows that $a \in (f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)$ since $f_1^j a = f_1^{j+1} c \in f_1^{j+1}\mathscr{M} \cap L$. This proves the implication $\Rightarrow$.

We now prove the implication $\Leftarrow$. Assume that $(f_1\mathscr{M} \cap L :_L f_2) \subseteq \cup_{j \geqslant 0} (f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)$. Let $a \in (f_1\mathscr{M} :_{\mathscr{M}} f_2)$ be an arbitrary element. Then $f_2 a = f_1 b$ for some element $b \in \mathscr{M}$. Since $\mathscr{M} = \cup_{e \geqslant 0} L_e$, there is an integer e such that $a \in L_e$.

Apply the functor $\mathbf{F}^e(-)$ to $(f_1\mathscr{M} \cap L :_L f_2) \subseteq \cup_{j \geqslant 0} (f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)$. Let $a \in (f_1\mathscr{M} :_{\mathscr{M}} f_2)$ imply that

$$(f_1^{p^e}\mathcal{M}\cap L_e :_{L_e} f_2^{p^e}) \subseteq \cup_{j\geqslant 0}(f_1^{(j+1)p^e}\mathcal{M}\cap L_e :_{L_e} f_1^{jp^e}).$$

The equation $f_2a = f_1b$ implies that $f_2^{p^e}f_1^{p^e-1}a = f_1^{p^e}f_2^{p^e-1}b$ and hence

$$f_1^{p^e-1}a \in (f_1^{p^e}\mathcal{M}\cap L_e :_{L_e} f_2^{p^e}) \subseteq \cup_{j\geqslant 0}(f_1^{(j+1)p^e}\mathcal{M}\cap L_e :_{L_e} f_1^{jp^e}).$$

Therefore, there is an integer ℓ and element $c \in \mathcal{M}$ such that

$$f_1^{(\ell+1)p^e-1}a = f_1^{\ell p^e}f_1^{p^e-1}a = f_1^{(\ell+1)p^e}c$$

which implies that

$$a \in (f_1^{(\ell+1)p^e}\mathcal{M} :_{\mathcal{M}} f_1^{(\ell+1)p^e-1}) \subseteq \cup_{j\geqslant 0}(f_1^{j+1}\mathcal{M} :_{\mathcal{M}} f_1^{j}).$$

This proves the implication $\Leftarrow$ and hence finishes the proof of our lemma.

Proof of Theorem 5.1. It follows from the exact sequence (5.0.3) that

$$\mathrm{Supp}(H^1(K^\bullet(f_1, f_2; \mathcal{M}))) = \mathrm{Supp}(H^1(\underline{f}; \Gamma)) \cup \mathrm{Supp}(\ker(\delta)).$$

Combining Theorem 5.2 and Lemma 5.3 completes the proof. □

Combining Theorems 4.1 and 5.1, the following result is immediate:

Theorem 5.4. *Let R be a Noetherian regular ring of prime characteristic p and $f_1, f_2 \in R$ form a regular sequence. Then, for every F-finite F-module,* $\mathrm{Supp}(H^i(K^\bullet(f_1, f_2; \mathcal{M}))$ *is Zariski-closed for each integer i.*

6 | THE SUPPORT OF $E_\infty^{\bullet,\bullet}$ WHEN $t = 2$

In this section, we prove that the support of $E_\infty^{i,j}$ is Zariski-closed for all integers i, j and the main theorem of this article: Theorem 6.5. Let R be a Noetherian commutative ring, $I = (g_1, \ldots, g_s)$ be an ideal, and $f_1, f_2 \in R$ be a regular sequence. Then the Koszul (co)complex $K^\bullet(\underline{f}; R)$ and the Čech complex $\check{C}^\bullet(\underline{g}; R)$ induce the double complex (2.0.1) introduced in Section 2. This double complex induces a spectral sequence whose $E_2^{\bullet,\bullet}$-page is as follows:

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H_I^j(R))) \Rightarrow H^{i+j}(T^\bullet).$$

Note that when $t = 2$ there is only one (potentially) nontrivial differential on the E_2-page:

$$d_2^{0,j} : E_2^{0,j} \to E_2^{2,j-1}.$$

Consequently

$$E_\infty^{1,j} = E_2^{1,j}, \quad E_\infty^{0,j} = E_3^{0,j} = \ker(d_2^{0,j}), \quad E_\infty^{2,j} = E_3^{2,j} = \mathrm{coker}(d_2^{0,j}). \tag{6.0.1}$$

We have seen in Section 5 that the support of $E_2^{1,j} = H^1(K^\bullet(f_1, f_2; H_I^j(R)))$ is Zariski-closed. It remains to show that both $\mathrm{Supp}(\ker(d_2^{0,j}))$ and $\mathrm{Supp}(\mathrm{coker}(d_2^{0,j}))$ are Zariski-closed. To this end, we begin with analyzing the construction of $d_2^{0,j}$.

Remark 6.1. We would like to recall the construction of $d_2^{0,j}$; the interested reader is referred to [18, 5.1.2] for more details. In order to cover the double complexes (2.0.1) and (3.0.4), we will consider a first quadrant double complex formed by the Koszul co-complex $K^\bullet(\underline{t}; R)$ on two elements t_1, t_2 and a finite complex $C^\bullet$ of R-modules (differentials in $C^\bullet$ will be denoted by $d_h^\bullet$):

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & C^0 & \to & C^1 & \to & C^2 & \to & \cdots & \to & C^s & \to & 0 \\
 & & \uparrow{\scriptstyle (-t_2\ t_1)} & & \uparrow{\scriptstyle (-t_2\ t_1)} & & \uparrow{\scriptstyle (-t_2\ t_1)} & & & & \uparrow{\scriptstyle (-t_2\ t_1)} & & \\
0 & \to & (C^0)^{\oplus 2} & \to & (C^1)^{\oplus 2} & \to & (C^2)^{\oplus 2} & \to & \cdots & \to & (C^s)^{\oplus 2} & \to & 0 \\
 & & \uparrow{\scriptstyle \binom{t_1}{t_2}} & & \uparrow{\scriptstyle \binom{t_1}{t_2}} & & \uparrow{\scriptstyle \binom{t_1}{t_2}} & & & & \uparrow{\scriptstyle \binom{t_1}{t_2}} & & \\
0 & \to & C^0 & \to & C^1 & \to & C^2 & \to & \cdots & \to & C^s & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array} \qquad (6.0.2)$$

Each element $[\eta] \in H^0(K^\bullet(t_1, t_2; H^j(C^\bullet))$ is an element $[\eta] \in H^j(C^\bullet)$ such that $(t_1[\eta], t_2[\eta]) = (0, 0) \in (H^j(C^\bullet))^{\oplus 2}$; equivalently $[\eta]$ can be represented by element $\eta \in C^j$ such that $d_h^j(\eta) = 0$ and there are elements $(\alpha_1, \alpha_2) \in (C^{j-1})^{\oplus 2}$ such that

$$d_h^{j-1}(\alpha_1) = t_1\eta \quad \text{and} \quad d_h^{j-1}(\alpha_2) = t_2\eta.$$

Consider $-t_2\alpha_1 + t_1\alpha_2 \in C^{j-1}$. Since

$$d_h^{j-1}(-t_2\alpha_1 + t_1\alpha_2) = -t_2 d_h^{j-1}(\alpha_1) + t_1 d_h^{j-1}(\alpha_2) = -t_2t_1\eta + t_1t_2\eta = 0,$$

the element $-t_2\alpha_1 + t_1\alpha_2 \in C^{j-1}$ represents an element $[-t_2\alpha_1 + t_1\alpha_2] \in H^{j-1}(C^\bullet)$. Then

$$d_2^{0,j}([\eta]) = \overline{[-t_2\alpha_1 + t_1\alpha_2]} \in E_2^{2,j-1} = H^2(K^\bullet(f_1, f_2; H^{j-1}(C^\bullet))) \cong \frac{H^{j-1}(C^\bullet)}{(t_1, t_2)H^{j-1}(C^\bullet)}.$$

For instance, the edge map in the spectral sequence associated with the double complex (3.0.4)

$$\varphi_{2,e}^{0,j} : H^0(K^\bullet(f_1^{p^e}, f_2^{p^e}; H^j(\check{C}^\bullet(\underline{g})_e)) \to H^2(K^\bullet(f_1^{p^e}, f_2^{p^e}; H^{j-1}(\check{C}^\bullet(\underline{g})_e))$$

can be described as follows. Each element $[\eta] \in H^0(K^\bullet(f_1^{p^e}, f_2^{p^e}; H^j(\check{C}^\bullet(\underline{g})_e))$ is an element $[\eta] \in H^j(\check{C}^\bullet(\underline{g})_e)$ such that $(f_1^{p^e}[\eta], f_2^{p^e}[\eta]) = (0, 0) \in (H^j(\check{C}^\bullet(\underline{g})_e))^{\oplus 2}$; equivalently $[\eta]$ can be represented by element $\eta \in \check{C}^j(\underline{g})_e$ such that $\delta^j(\eta) = 0$ and there are elements $\alpha_1, \alpha_2 \in \check{C}^{j-1}(\underline{g})_e$ such that

$$\delta^{j-1}(\alpha_1) = f_1^{p^e}\eta \quad \text{and} \quad \delta^{j-1}(\alpha_2) = f_2^{p^e}\eta.$$

Consider $-f_2^{p^e}\alpha_1 + f_1^{p^e}\alpha_2 \in C_e^{j-1}$. Since

$$\delta^{j-1}(-f^{p^e}2\alpha_1 + f_1^{p^e}\alpha_2) = -f_2^{p^e}\delta^{j-1}(\alpha_1) + f_1^{p^e}\delta^{j-1}(\alpha_2) = -f_2^{p^e}f_1^{p^e}\eta + f_1^{p^e}f_2^{p^e}\eta = 0$$

the element $-f_2^{p^e}\alpha_1 + f_1^{p^e}\alpha_2 \in \check{C}^{j-1}(\underline{g})$ represents an element $[-f_2^{p^e}\alpha_1 + f_1^{p^e}\alpha_2] \in H_I^{j-1}(R)$. Then

$$\varphi_{2,e}^{0,j}([\eta]) = \overline{[-f_2^{p^e}\alpha_1 + f_1^{p^e}\alpha_2]} \in H^2(K^\bullet(f_1^{p^e}, f_2^{p^e}; H^{j-1}(\check{C}_e^\bullet))) \cong \frac{H^{j-1}(\check{C}^\bullet(\underline{g})_e)}{(f_1^{p^e}, f_2^{p^e})H^{j-1}(\check{C}^\bullet(\underline{g})_e)}. \tag{6.0.3}$$

To ease notation, for the rest of this section we will denote the Čech complex $\check{C}^\bullet(\underline{g})$ by $\check{C}^\bullet$ and its eth truncation $\check{C}^\bullet(\underline{g})_e$ by $\check{C}_e^\bullet$.

Recall that the double complex $\mathbf{D}_0$ induces the spectral sequence (3.0.5):

$$E_{2,0}^{i,j} := H^i(K^\bullet(\underline{f}; H^j(\check{C}_0^\bullet))) \Rightarrow H^{i+j}(T_0^\bullet)$$

with the differentials

$$\varphi_{2,0}^{i,j} : H^0(K^\bullet(\underline{f}; H^j(\check{C}_0^\bullet))) \to H^2(K^\bullet(\underline{f}; H^{j-1}(\check{C}_0^\bullet))).$$

Let $K_0^j \subseteq \ker(\delta_0^j) \subseteq \check{C}_0^j$ be the submodule whose image in $H^j(\check{C}^\bullet)$ is the kernel of $\varphi_2^{0,j}$, where δ_0^j denotes the jth differential in $\check{C}_0^\bullet$. Note that

(1) K_0^j is a finitely generated R-module since $\check{C}_0^j$ is so;

(2) the image of $H^j(\check{C}_0^\bullet)$ in $H_I^j(R)$ is isomorphic to $\frac{\ker(\delta_0^j)}{\ker(\delta_0^j)\cap \operatorname{image}(\delta^{j-1})}$, where δ^j denotes the jth differential in $\check{C}^\bullet$; this is contained in (3.0.3).

First we treat $\operatorname{Supp}(E_\infty^{0,j})$ which is $\operatorname{Supp}(\ker d_2^{0,j})$ (6.0.1) and we begin with the following lemma.

Lemma 6.2. *Let R be a Noetherian regular ring of prime characteristic p. Let $\varphi_{2,e}^{0,j}$ be defined as in (6.0.3). Let K_e^j be the submodule of $\ker(\delta_e^j) \subseteq \check{C}_e^j$ whose image in $H^j(\check{C}_e^\bullet)$ is the kernel of $\varphi_{2,e}^{0,j}$. Let $\theta : \mathbf{F}^e(\check{C}_0^j) \xrightarrow{\sim} \check{C}_e^j$ denote the isomorphism in Proposition 3.3. Then*

$$\theta(\mathbf{F}^e(K_0^j)) = K_e^j.$$

Proof. This follows from the commutative diagram below and the fact $R^{(e)}$ is a faithfully flat R-module.

$$
\begin{array}{ccc}
R^{(e)} \otimes (\check{C}_0^{j-1} \oplus \check{C}_0^{j-1}) & \xrightarrow{\sim} & \check{C}_e^{j-1} \oplus \check{C}_e^{j-1} \\
\Big\downarrow {\scriptstyle \mathbf{1}\otimes(\delta_0^{j-1}\oplus\delta_0^{j-1})} & & \Big\downarrow {\scriptstyle \delta_e^{j-1}\oplus\delta_e^{j-1}} \\
R^{(e)} \otimes (\check{C}_0^{j} \oplus \check{C}_0^{j}) & \xrightarrow{\sim} & \check{C}_e^{j} \oplus \check{C}_e^{j} \\
\Big\uparrow {\scriptstyle \mathbf{1}\otimes\binom{f_1}{f_2}} & & \Big\uparrow {\scriptstyle \mathbf{1}\otimes\binom{f_1^{p^e}}{f_2^{p^e}}} \\
R^{(e)} \otimes \check{C}_0^{j} & \xrightarrow{\sim} & \check{C}_e^{j}
\end{array}
$$

□

Theorem 6.3. *Let R be a Noetherian regular ring of prime characteristic p and let $E_2^{\bullet,\bullet}$ be the E_2-page of the spectral sequence associated with the double complex (2.0.1). Then* $\ker d_2^{0,j} = 0$ *if and only if $K_0^j \subseteq \operatorname{image}(\delta^{j-1})$; that is,*

$$\operatorname{Supp}(E_\infty^{0,j}) = \operatorname{Supp}(\ker d_2^{0,j}) = \operatorname{Supp}\left(\frac{K_0^j}{K_0^j \cap \operatorname{image}(\delta^{j-1})}\right). \tag{6.0.4}$$

In particular, $\operatorname{Supp}(E_\infty^{0,j}) = \operatorname{Supp}(\ker d_2^{0,j})$ *is Zariski-closed.*

Proof. The second statement follows from (6.0.4) since K_0^j is finitely generated.

To complete the proof, it remains to show that $\ker d_2^{0,j} = 0$ if and only if $K_0^j \subseteq \operatorname{image}(\delta^{j-1})$.

Assume that $d_2^{0,j}$ is injective and $[\eta] \in K_0^j$. One needs to show that $[\eta] \in \operatorname{image}(\delta^{j-1})$. Since $[\eta]$ belongs to K_0^j, its image in $H^j(\check{C}_0^\bullet)$ must belong in $\ker(\varphi_{2,0}^{0,j})$. It follows that the image of $[\eta]$ in $H_I^j(R)$ must belong in $\ker(d_2^{0,j})$. Since $d_2^{0,j}$ is injective, the image of $[\eta]$ in $H_I^j(R)$ must be $[0]$, which implies that $[\eta] \in \operatorname{image}(\delta^{j-1})$. This proves the "if" statement.

Assume that $K_0^j \subseteq \operatorname{image}(\delta^{j-1})$; that is, if $\varphi_{2,0}^{0,j}([\eta]) = [0]$, then $\eta \in \operatorname{image}(\delta^{j-1})$ (equivalently, the image $[\eta]$ of η in $H_I^j(R)$ is zero). Note it follows from Lemma 6.2 that

$$K_e^j \cong \mathbf{F}^e(K_0^j) \subseteq \mathbf{F}^e(\operatorname{image}(\delta^{j-1})) \cong \operatorname{image}(\delta^{j-1}),$$

where the last isomorphism follows from the fact that δ^{j-1} is a differential in the Čech complex and hence an F-module morphism.

Let $[\tau]$ be an element in $\ker(d_2^{0,j})$, it remains to show that $[\tau] = [0] \in H_I^j(R)$. Since $[\tau] \in \ker(d_2^{0,j})$, there are elements $\tau \in \check{C}^j$ and $\alpha_1, \alpha_2 \in \check{C}^{j-1}$ such that

$$\delta^{j-1}(\alpha_1) = f_1\tau,\ \delta^{j-1}(\alpha_2) = f_2\tau, \text{ and } d_2^{0,j}([\tau]) = \overline{[-f_2\alpha_+ f_1\alpha_2]} \in (f_1, f_2)H_I^{j-1}(R).$$

Since there are finitely many cohomology classes involved, there exists an integer e such that $\tau \in \check{C}_e^j$, $\alpha_1, \alpha_2 \in \check{C}_e^{j-1}$, and that $d_2^{0,j}([\tau])$ can be represented by an element in $(f_1, f_2)H^{j-1}(\check{C}_e^\bullet)$. We will fix one such e and we consider the double complex (3.0.4) for this integer e. It follows that

$$\delta^{j-1}(f_1^{p^e-1}\alpha_1) = f_1^{p^e}\tau \text{ and } \delta^{j-1}(f_2^{p^e-1}\alpha_2) = f_2^{p^e}\tau.$$

According to the description of the edge map (6.0.3) associated with the double complex (3.0.4):

$$\begin{aligned}\varphi^{0,j}_{2,e}([\tau]) &= \overline{[-f_2^{p^e} f_1^{p^e-1}\alpha_1 + f_1^{p^e} f_2^{p^e-1}\alpha_2]}\\ &= (f_1^{p^e-1} f_2^{p^e-1})\overline{[-f_2\alpha_+ f_1\alpha_2]}\\ &\in (f_1^{p^e-1} f_2^{p^e-1})(f_1, f_2)H^{j-1}(\check{C}^\bullet_e)\\ &\in (f_1^{p^e}, f_2^{p^e})H^{j-1}(\check{C}^\bullet_e).\end{aligned}$$

That is $[\tau]$ belongs in K^j_e and consequently $[\tau] \in K^j_e \subseteq \mathrm{image}(\delta^{j-1})$. Thus, the image of $[\tau]$ in $H^j_I(R)$ is zero. This shows that, if $K^j_0 \subseteq \mathrm{image}(\delta^{j-1})$, then $d^{0,j}_2$ is injective, which completes the proof. □

Theorem 6.4. *Let R be a Noetherian regular ring of prime characteristic p and let $E_2^{\bullet,\bullet}$ be the E_2-page of the spectral sequence associated with the double complex (2.0.1). Let $H \subseteq H^{j-1}_I(R)$ be the submodule generated by elements that can be represented by elements in $\check{C}^{j-1}_0$. Let $L \subseteq H^{j-1}_I(R)$ be the submodule whose image in $H^{j-1}_I(R)/(f_1, f_2)H^{j-1}_I(R)$ is $\mathrm{image}(d^{0,j}_2)$. Then $d^{0,j}_2$ is surjective if and only if $H \subseteq L$; that is,*

$$\mathrm{Supp}(E^{2,j-1}_\infty) = \mathrm{Supp}(\mathrm{coker}\, d^{0,j}_2) = \mathrm{Supp}\left(\frac{H}{H \cap L}\right). \tag{6.0.5}$$

In particular, $\mathrm{Supp}(E^{2,j-1}_\infty) = \mathrm{Supp}(\mathrm{coker}\, d^{0,j}_2)$ *is Zariski-closed.*

Proof. Since H is finitely generated (3.0.3), the Zariski-closedness follows from the "if and only if" statement.

If $d^{0,j}_2$ is surjective, then $L = H^{j-1}_I(R)$ and hence $H \subseteq L$.

Assume that $H \subseteq L$. Then $\mathbf{F}^e(H) \subseteq \mathbf{F}^e(L)$ for each e since $\mathbf{F}$ is an exact functor. Note that $\mathbf{F}^e(H)$ is the submodule of $H^{j-1}_I(R)$ generated by elements that can be represented by elements in $\check{C}^{j-1}_e$ and that $\mathbf{F}^e(L)$ is the submodule of $H^{j-1}_I(R)$ whose image in $H^{j-1}_I(R)/(f_1^{p^e}, f_2^{p^e})H^{j-1}_I(R)$ is $\mathrm{image}(\varphi^{0,j}_{2,e})$.

Let $[\eta]$ be an arbitrary element in $H^{j-1}_I(R)/(f_1, f_2)H^{j-1}_I(R)$. Pick an element η_e in $\check{C}^{j-1}_e$ whose image in $H^{j-1}_I(R)/(f_1, f_2)H^{j-1}_I(R)$ is $[\eta]$. Then $[\eta_e] \in H^{j-1}_I(R)$ belongs to $\mathbf{F}^e(H)$. Hence $[\eta_e] \in \mathbf{F}^e(L)$; that is, there are $\tau_e \in \check{C}^j_e$, $\alpha_{1,e}, \alpha_{2,e} \in \check{C}^{j-1}_e$ and $\beta_{1,e}, \beta_{2,e} \in \ker(\delta^j_e)$ such that

$$\delta^j_e(\tau_e) = 0,\ \delta^{j-1}_e(\alpha_{1,e}) = f_1^{p^e}\tau_e,\ \delta^{j-1}_e(\alpha_{2,e}) = f_2^{p^e}\tau_e$$

and that

$$\begin{aligned}[\eta_e] &= \varphi^{0,j}_{2,e}([\tau_e])\\ &= \overline{[-f_2^{p^e}\alpha_{1,e} + f_1^{p^e}\alpha_{2,e}]} + f_1^{p^e}\beta_{1,e} + f_2^{p^e}\beta_{2,e}\\ &= \overline{[-f_2(f_2^{p^e-1}\alpha_{1,e}) + f_1(f_1^{p^e-1}\alpha_{2,e})]} + f_1(f_1^{p^e-1}\beta_{1,e}) + f_2(f_2^{p^e-1}\beta_{2,e}).\end{aligned}$$

Set $\widetilde{\tau} = f_1^{p^e-1} f_2^{p^e-1} \tau_e$, $\widetilde{\alpha}_1 = f_2^{p^e-1}\alpha_{1,e}$ and $\widetilde{\alpha}_2 = f_1^{p^e-1}\alpha_{2,e}$. Then

$$\delta_e^j(\widetilde{\tau}) = 0,\ \delta_e^{j-1}(\widetilde{\alpha}_1) = f_1\widetilde{\tau},\ \delta_e^{j-1}(\widetilde{\alpha}_2) = f_2\widetilde{\tau}$$

and

$$\begin{aligned}[\eta_e] &= \overline{[-f_2(f_2^{p^e-1}\alpha_{1,e}) + f_1(f_1^{p^e-1}\alpha_{2,e})]} + f_1(f_1^{p^e-1}\beta_{1,e}) + f_2(f_2^{p^e-1}\beta_{2,e})\\ &= \overline{[-f_2\widetilde{\alpha}_1 + f_1\widetilde{\alpha}_2]} + f_1(f_1^{p^e-1}\beta_{1,e}) + f_2(f_2^{p^e-1}\beta_{2,e})\\ &= d_2^{0,j}(\widetilde{\tau}).\end{aligned}$$

This proves that $[\eta_e]$ is in the image of $d_2^{0,j}$. This completes the proof. □

Combining Theorems 2.4, 5.4, 6.3, and 6.4, the following theorem is immediate:

Theorem 6.5. *Let R be a Noetherian regular ring of prime characteristic p. If $f_1, f_2 \in R$ form a regular sequence in R, then*

$$\operatorname{Supp}\left(H_I^j\left(\frac{R}{(f_1, f_2)}\right)\right)$$

is Zariski-closed for each ideal I and each integer j.

The following corollary is immediate.

Corollary 6.6. *Let R be a Noetherian commutative ring of prime characteristic p that has finitely many isolated singular points. Let $f_1, f_2 \in R$ be a regular sequence. Then $H_I^j(R/(f_1, f_2))$ is Zariski-closed for each integer j and each ideal I.*

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