Koszul Complexes, Local Cohomology, Universal Resolutions, and Cohomological Support Varieties

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February 26, 2026

LIBERAL ARTS AND SCIENCES

Mathematics, Statistics, and Computer Science

A note about this defense

Green slides are for a more general audience

White slides are for fellow students and my committee

Overview of the defense

This thesis is broken into three parts:

- Results regarding the support of local cohomology modules,
- Survey on Golod rings, Massey products and A_∞ structures,
- Results regarding the cohomological support varieties of certain monomial ideals.

We will spend most of our time discussing the last of these, reserving some time at the end for the first two.

The Koszul Complex: Our One Connecting Thread

Consider a collection of elements $\boldsymbol{f} = (f_1, \ldots, f_c)$ of a ring R. The *Koszul complex* is the complex

$$\bigotimes_{1 \leq i \leq c} 0 \to R \xrightarrow{f_i} R \to 0.$$

This can be written as

$$0 \to \bigwedge^{c} R^c \to \bigwedge^{c-1} R^c \to \cdots \to \bigwedge^{1} R^c \to \bigwedge^{0} R^c \to 0$$

with differential

$$a_1 \wedge \cdots \wedge a_n \mapsto \sum_i (-1)^{i+1} a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n.$$

Some facts about Koszul complexes

If R is a field like $\mathbb{R}$, then each $\bigwedge^i R^c$ is a vector space, with one basis element for each set of i elements of $\{1, \ldots, c\}$, the arrows denote linear maps, and the fact that we call this a *complex* means that the composition of two of these linear maps is zero.

If R is the set of all polynomials in variables x, y, z with coefficients in $\mathbb{R}$, and $\boldsymbol{f} = (x, y)$, then the Koszul complex is

$$0 \to R \xrightarrow{\binom{y}{x}} R^2 \xrightarrow{(-x\ y)} R \to 0.$$

The pairs (p, q) of polynomials in the middle which are taken to 0 are exactly those pairs coming from our copy of R on the left. Our complex is *exact* (no *homology*) at this entry. If our complex is *exact* at every entry except the last, and each entry is $R^{\text{some value}}$, we call it a *resolution* of the cokernel (in this case $R/(x, y) \cong \mathbb{R}$) over R.

DG algebras

A *DG algebra* is a unital algebra with a chain complex structure:

- $A = \bigoplus_{i \in \mathbb{Z}} A_i$ with differential $\partial = \{\partial_i : A_i \to A_{i-1} \mid i \in \mathbb{Z}\}$ with $\partial^2 = 0$,
- If $a \in A_i$ and $b \in A_j$ then $ab \in a_{i+j}$.

Our structure must be associative and transitive. We also add some additional constraints, which are sometimes in the definition but sometimes merit the name *commutative* DG algebra *concentrated in non-negative degree*:

- If $a \in A_i$ and $b \in A_j$ then $ba = (-1)^{ij} ab$.
- $A_i = 0$ for $i < 0$.

The Koszul complex is a DG algebra; we need only assign a multiplication structure:

$$(a_1 \wedge \cdots \wedge a_m)(b_1 \wedge \cdots \wedge b_n) = a_1 \wedge \cdots \wedge a_m \wedge b_1 \wedge \cdots \wedge b_n$$

DG modules

A DG module is a chain complex U with a multiplication action $A \otimes U \to U$ from a DG algebra satisfying the Leibniz rule:

$$\partial(au) = \partial(a)u + (-1)^{|a|} a\partial(u)$$

Can we get DG algebra structures on a given resolution?

Not always, but sometimes!

Theorem (see Avramov '98)

Let k be a field, and Q be the polynomial ring $k[s_1, s_2, s_3, s_4]$ with the usual grading, or the power series ring $k[\![s_1, s_2, s_3, s_4]\!]$. There exists no DG algebra structure on the minimal Q-free resolution U of the Cohen-Macaulay residue ring $R = Q/I$ where

$$I' = I + (s_1 s_3^6, s_2^7, s_2^6 s_4, s_3^7).$$

Theorem (see Avramov '98)

If A is a projective resolution of a Q-module R, such that $A_0 = Q$ and $A_n = 0$ for $n \geq 4$, then A has a structure of DG algebra.

Universal resolutions

Universal resolutions are "free resolutions over a ring that are defined in a uniform way for all finitely generated modules" (Briggs, Grifo, Pollitz '25).

Let $R = Q/I$, A a DG algebra resolving R over Q, $\tilde{A} = \operatorname{coker}(Q \to A)$, F an A resolution of an R-module M. The *bar construction* is the totalization of

$$\cdots \to A \otimes_Q \tilde{A} \otimes_Q \tilde{A} \otimes_Q F \to A \otimes_Q \tilde{A} \otimes_Q F \to A \otimes_Q F \to F \to 0,$$

$$\begin{aligned}\partial := a \otimes \tilde{a}_1 \otimes \cdots \otimes \tilde{a}_n \otimes f \mapsto (a\tilde{a}_1) \otimes \tilde{a}_2 \otimes \cdots \otimes \tilde{a}_n \otimes f \\ + \sum_{i=1}^{n-1} (-1)^{i-1} a \otimes \tilde{a}_1 \otimes \cdots \otimes (\tilde{a}_i \tilde{a}_{i+1}) \otimes \cdots \otimes a_n \otimes f \\ + (-1)^n a \otimes \tilde{a}_1 \otimes \cdots \otimes \tilde{a}_{n-1} \otimes (\tilde{a}_n f).\end{aligned}$$

The bar construction tensored over A with R is an R-resolution of M.

Our results on cohomological support varieties

We provide the relevant results. First is a method for more efficiently calculating the cohomological support variety of a monomial ideal:

Theorem (G ’26)

The cohomological support variety of an equigenerated monomial ideal with a minimal regular presentation with n generators is expressible as the homology of a chain complex of vector spaces with total dimension 2^n.

Our results on cohomological support varieties (continued)

The second is a theoretical verification and extension of a computed result by Briggs, Grifo and Pollitz '24:

Theorem (G '26)

The cohomological support variety of the ideal $(x_1x_2, x_2x_3, \dots, x_6x_1)$ in $k[x_1, \dots, x_6]$ is $\mathcal{V}(a_1a_3a_5 + a_2a_4a_6)$ where a_i are the values of the coordinates corresponding to each variable.

Furthermore, the cohomological support variety of the ideal $(x_1x_2, x_2x_3, \dots, x_{10}x_1)$ in $k[x_1, \dots, x_{10}]$ is $\mathcal{V}(a_1a_3a_5a_7a_9 + a_2a_4a_6a_8a_{10})$ where a_i are as above.

The Avramov-Buchweitz resolution

The Avramov-Buchweitz resolution is a universal E-resolution of an $R = Q/I$ module M:

- First, choose a minimal generating set $\boldsymbol{f}$ of I,
- Second, let E be the DG algebra $Q[\xi_1, \ldots, \xi_n \mid \partial\xi_i = f_i]$ (the Koszul complex!),
- Third, choose a Q-module resolution F of M which is also an E-module such that the augmentation is a map of DG E-modules (a *Koszul resolution*).

The Avramov-Buchweitz resolution is given by defined as $E \otimes_Q \Gamma \otimes_Q F$ where Γ is a polynomial ring with generators χ_j of homological degree -2, with a differential which can be written as

$$\partial_E \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \partial_F + \sum_j \left(1 \otimes \chi_j \otimes \xi_j - \xi_j \otimes \chi_j \otimes 1\right).$$

Tensoring over E with R is a resolution when R is a complete intersection.

Minimal regular presentations

Definition

R has a *minimal regular presentation* Q/I if one of the following holds:

- $\widehat{R} = Q/I$ with Q a regular local ring and I in the square of the maximal ideal of Q (the *local case*),
- $R = Q/I$ where Q is a positively graded polynomial algebra over a field and I an ideal generated by homogeneous forms of degree at least 2 (the *graded case*).

In both cases, there is a residue field k to speak of, which will be necessary for the cohomological support varieties we consider. We consider such R henceforth.

Support varieties

The Avramov-Buchweitz resolution is a Γ module. Thus, taking Hom to some module N yields a Γ^* structure on $\operatorname{Ext}_E(M, N)^\bullet$.

When R has a minimal regular presentation, we will consider the support of $\operatorname{Ext}_E^\bullet(M, k)$ as a projective variety. This is known as the *cohomological support variety* of R, defined originally by Avramov and Buchweitz in '00, though inspection of these χ_j goes back to Gulliksen '74.

Theorem (Pollitz '21)

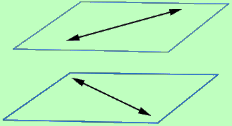
The CSV of $R = Q/I$ is empty if and only if R is a complete intersection.

What is a complete intersection?

In 3-dimensional space, any polynomial which has a solution defines a 2-dimensional (*codimension 1*) surface. If I want to make a codimension c object, I would hope that I can just take the intersection of c objects of codimension 1.

You can't define two polynomial equations in x, y, z such that the points satisfying both are exactly the points on one of the two lines below (try it!), so this object isn't a CI.

Type check: An example of a *ring* which is a CI (which is what we usually work with here) is the set of all polynomial functions on these two lines.



Realizability and our contribution

The *realizability problem* for CSVs asks what values CSVs of various rings and modules can take on, in order to determine "how much" information the support variety can encode.

We consider first a more computable example, that given by monomial ideals.

Theorem (Briggs, Grifo, Pollitz '25)

If I is minimally generated by 5 monomials, the CSV of R is a union of coordinate subspaces of $\mathbb{P}^4_k$.

Realizability and our contribution (continued)

They also computationally verified the following

Computation (Briggs, Grifo, Pollitz '25)

The CSV of $(x_1x_2, x_2x_3, \ldots, x_6x_1)$ *in* $k[x_1, \ldots, x_6]$ *is* $\mathcal{V}(a_1a_3a_5 + a_2a_4a_6)$.

Our innovation was finding a more computationally efficient and human-executable method for finding CSVs in certain cases:

Theorem (G '26)

The cohomological support variety of an equigenerated monomial ideal with a minimal regular presentation is expressible as the homology of a chain complex of vector spaces with total dimension 2^n.

The aforementioned chain complex can be expressed as a double complex with rows the reduced cohomology of simplicial complexes with at most n points.

The Taylor resolution

The monomial ideal resolution used in these calculations is the *Taylor resolution* $T(\boldsymbol{f})$:

- $(T(\boldsymbol{f}))_i : \{b'_J : J \subseteq [n] \text{ with } |J| = i\}$,
- $\partial(b'_J) = \sum_{i=1}^{s} (-1)^{i-1} \frac{f_J}{f_{J\setminus\{j_i\}}} b'_{J\setminus\{j_i\}} \quad \text{with } J = \{j_1 < \cdots < j_s\}$.

We let b_J denote the basis of $T(\boldsymbol{f}) \otimes k$. This is a Koszul resolution, with b_j acting as ξ_j.

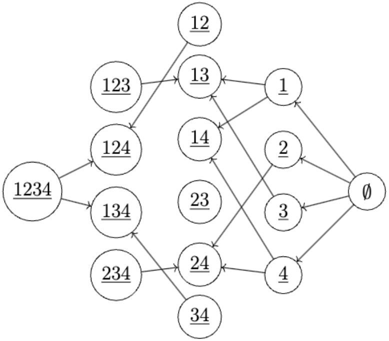
Theorem (well-known)

To determine the support variety of R, we can take the homology of the vector-space-with-automorphism $\widehat{\mathscr{C}}_{E_a}(T(\boldsymbol{f}))'$ whose bases can be identified with $T(\boldsymbol{f}) \otimes k$:

- $(\widehat{\mathscr{C}}_{E_a}(T(\boldsymbol{f}))')_i : \{b_J : J \subseteq [n] \textit{ with } |J| = i\}$,
- $\partial(b_J) = \sum_{i=1}^{s} (-1)^{i-1} \frac{f_J}{f_{J\setminus\{j_i\}}} b_{J\setminus\{j_i\}} \quad \textit{with } J = \{j_1 < \cdots < j_s\}, f_J = f_{j_i} f_{J\setminus\{j_i\}}, u \in Q^*$
  $+ \sum_{i \notin J} \operatorname{sgn}(i, j_1, \ldots, j_s)\, b_{J\cup\{i\}} \quad \textit{for } J = \{j_1 < \cdots < j_s\},\ \gcd(f_i, f_j) = 1\ \forall j \in J$

A picture of $\widehat{\mathscr{C}}_{E_a}(T(f))'$

$\underline{123}$ is used as shorthand for $\{1, 2, 3\}$



Our contribution

Partitioning our b_J by the LCM of $\{f_j \mid j \in J\}$, partitions $T(\boldsymbol{f}) \otimes k$ into subcomplexes $T_{M_J}(\boldsymbol{f})$, where M_J is the largest set with a given LCM. Furthermore, these subcomplexes have homologies given by those of simplicial complexes:

Theorem (G '26)

$T_{M_J}(\boldsymbol{f})$ shares a differential structure (up to a shift) with the reduced cochain complex of the simplicial complex given by the subsets which can be removed from M_J without affecting their LCM.

Theorem (G '26)

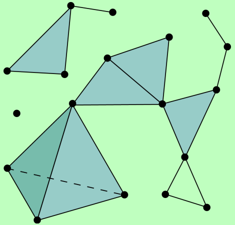
In some cases including the equigenerated monomial ideal case, our $T_{M_J}(\boldsymbol{f})$ subcomplexes can be shifted such that our differential of $\widehat{\mathscr{C}_{E_a}}(T(\boldsymbol{f}))'$ has degree -1, and the resulting complex can be written as a double complex with rows direct sums of terms $T_{M_J}(\boldsymbol{f})$.

A sketch of our double complex

$$
\begin{array}{ccccccc}
 & & \emptyset & & & & \\
 & & \downarrow & & & & \\
 & & 1,2,3,4 & & & 12,34 & \\
 & & \downarrow & & & \downarrow & \\
234,123 & \longrightarrow & 13,14,24 & & 1234 \longrightarrow & 124,134 & \quad 23
\end{array}
$$

Simplicial Complexes

A *simplicial complex* can be thought of as a collection of triangles / tetrahedra of various dimensions:



The *differential* of some triangle is the set of its boundaries, each with some sign. We are saying that our map / function which gives our support varieties has a structure which comes from this one, and if we can describe these (well-studied) simplicial complex differentials we can much more easily describe the maps we care about.

An example of our double complex: edge ideal of a 6-cycle

Let our ideal be $(x_1x_2, x_2x_3, \ldots, x_6x_1)$. Our double complex is as follows, where Σ' indicates our shift, σ is our rotation permutation, and $\boldsymbol{f}$ is omitted:

$$
\begin{array}{c}
T_{\varnothing}^{\Sigma'} \\
\downarrow \\
\bigoplus \langle \sigma \rangle T_{\underline{1}}^{\Sigma'} \\
\downarrow \\
\bigoplus \langle \sigma \rangle T_{\underline{123}}^{\Sigma'} \oplus \bigoplus \langle \sigma^2 \rangle T_{\underline{14}}^{\Sigma'} \\
\downarrow \\
T_{\underline{123456}}^{\Sigma'}
\end{array}
$$

We can perform spectral sequence calculations to leverage our understanding of our horizontal maps.

Computer-assisted proofs

In the equigenerated monomial ideal in a polynomial ring over $\mathbb{Q}$ case, we wrote a program which quickly calculates the CSV using this chain complex.

Computation (G '26)

The cohomological support variety of the ideal $(x_1x_2, x_2x_3, \ldots, x_{14}x_1)$ *in* $\mathbb{Q}[x_1, \ldots, x_{14}]$ *is* $\mathcal{V}(a_1a_3 \cdots a_{13} + a_2a_4 \cdots a_{14})$ *where* a_i *are the values of the coordinates corresponding to each variable.*

Computer-assisted proofs (continued)

CSVs are a function only of the divisibilities and coprimalities between sets of elements of $\boldsymbol{f}$, which allows us to consider a finite set of equigenerated monomial ideals. By eliminating some ideals with full support, we can construct a cone whose generators yield every possible CSV:

Computation (G '26)

Every equigenerated monomial ideal in a polynomial ring over $\mathbb{Q}$ with 6 generators yields a support variety which is a union of coordinate subspaces or $\mathcal{V}(a_1a_3a_5 + a_2a_4a_6)$ up to order.

Support of local cohomology modules of complete intersections

This paper calculated support of local cohomology modules of regular sequences.

Theorem (G, Zhang ’26)

Let S be a Noetherian regular ring of prime characteristic p and f_1, f_2 be a regular sequence in S. Set $R = S/(f_1, f_2)$. Then $\operatorname{Supp}(H^j_I(R))$ is Zariski-closed for each integer j and each ideal I.

Support of local cohomology modules of complete intersections (continued)

Theorem (G, Zhang '26)

Let R be a Noetherian ring, $I = (g_1, \ldots, g_t)$ be an ideal, and $f_1, \ldots, f_c$ be a sequence of elements in R. Assume that

1. *The entries of the E_∞ page of the spectral sequence*

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H_I^j(R)) \Rightarrow H^{i+j}(T^\bullet),$$

   where $T^\bullet$ denotes the relevant total complex, are Zariski-closed, and that

2. *$f_1, \ldots, f_c$ form a regular sequence in R.*

Then $\mathrm{Supp}(H_I^k(R/(f_1, \ldots, f_c)))$ is Zariski-closed for each integer k.

Massey products and A_∞ algebras via Golodity

We have a chapter which is primarily a survey on the various definitions of Golodity, how Golodity can be used as a lens to understand how to translate Massey product-based proofs into A_∞-algebra based proofs, and some existing results which embody some limitations to these translations which I think are important for the community to be aware of when choosing how to characterize information which can be expressed in either form.

Thank you!