Koszul Complexes, Local Cohomology, Universal Resolutions, and Cohomological Support Varieties

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THESIS

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Furthermore, these constraints made infeasible the avoidance of the following diversions from the thesis manual not implied by the above:

- There are no dots following entries in the table of contents or the table of figures.
- Subsections are not underlined.

DEDICATION

This thesis is dedicated to Anna, without whom I would be stuck in a well.

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Thank you to my advisor Kevin Tucker, Wenliang Zhang for many discussions culminating in the work begetting Chapter II and Chapter III, which we co-authored, Eloísa Grifo and Josh Pollitz for valuable conversations, especially regarding Chapter IV, Chapter V, Chapter VI, and Chapter VII, and Izzet Coskun, Lawrence Ein, and Maya Banks for their valuable input on the final document.

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The water may deepen, and the storms may roll in, but I am never alone.

Standing on your shoulders, I can see you eye to eye.

MCG

CONTRIBUTION OF AUTHORS

Chapter II and Chapter III together comprise a reproduction of [GZ25], which is joint work with Wenliang Zhang. Its form here has undergone only clerical and typesetting changes.

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LIST OF ABBREVIATIONS

CDG: commutative differential graded
DG: differential graded
GCD: greatest common denominator
LCM: least common multiple

SUMMARY

We will explore three fields which utilize the Koszul complex: local cohomology, Golodity, and the cohomological support variety. Chapter I outlines the work. Chapter II and Chapter III explore the support of local cohomology modules, showing in particular the Zariski closure of certain local cohomology modules of ideals generated by regular sequences.

In Chapter IV, Chapter V, Chapter VI, and Chapter VII we explore various universal resolutions. Chapter IV provides relevant background for the following three sections, comprising an overview of DG algebras and universal resolutions. Chapter V is largely a survey, concerning the relationship between some relatively common conditions on module resolutions, namely Golodity and formality, and A_∞ algebras. We go over some of the definitions of formality and Golodity in terms of A_∞ algebras while providing some explanations regarding the bridges between them in the language of Massey products. We explicitly describe a relatively well-known relationship between Golodity and formality in a way slightly different from descriptions seen elsewhere using this framework.

In Chapter VI and Chapter VII we discuss the cohomological support variety, and specifically its construction when resolving a quotient of a monomial ideal over a polynomial ring. We provide a construction for calculating support varieties of monomial ideals which is slightly more computationally manageable than any known construction. We use this to manually verify a computation of the cohomological support variety of the edge ideal on a cycle with 6 vertices. We also computationally verify a classification of all cohomological support varieties of equigenerated monomial ideals, as well as a description of the cohomological support varieties of edge ideals with ten and fourteen vertices.

I

INTRODUCTION

The Koszul complex is a well-known chain complex constructed from a collection of ring elements. In particular, for R a ring and $\boldsymbol{f}$ some sequence of elements (historically, a regular sequence), the Koszul complex on $\boldsymbol{f} = (f_i)$ is the tensor of the two-entry complexes given by the multiplication-by-f_i maps on R. The Koszul complex is integral to a number of fields and constructions. When $\boldsymbol{f}$ is a regular sequence it resolves $R/\boldsymbol{f}$, and it is the simplest and most common example of a differential graded (DG) algebra. We will explore three fields which utilize the Koszul complex: local cohomology, Golodity, and the cohomological support variety. It is important to note that our notation for Chapter II and Chapter III will be self-consistent, as will that for Chapter IV, Chapter V, Chapter VI, and Chapter VII, but these may not always be self-consistent between each other. Every ring in this work is Noetherian.

In Chapter II and Chapter III we will explore the support of local cohomology modules. Let R be a Noetherian commutative ring and I be an ideal. Let Γ_I denote the I-torsion functor defined via:

$$\Gamma_I(M) = \{z \in M \mid I^t z = 0 \text{ for some integer } t\}; \quad \Gamma_I(M \xrightarrow{f} N) = \Gamma_I(M) \xrightarrow{f|_{\Gamma_I(M)}} \Gamma_I(N).$$

It turns out that Γ_I is left-exact; the j-th local cohomology of an R-module M, denoted by $H^j_I(M)$, is defined as $\mathbb{R}^j\Gamma_I(M)$; that is

$$H^j_I(M) \cong H^j(0 \to Q^\bullet)$$

where $0 \to M \to Q^\bullet$ is an injective resolution of M. It can be calculated by a Čech complex; *cf.* Section 1 of Chapter II for details.

Since the theory of local cohomology was introduced in [Gro68], the study of finiteness properties of these modules, as well as their vanishing, has become an active research topic. The interested reader is referred to [Hun92] for a list of inspiring open questions on vanishing and finiteness properties of local cohomology modules. One of these question asks whether the set of associated primes of $H^j_I(R)$ is finite for each integer j and each ideal I in R. Some positive answers are known: when R is a regular ring of equi-characteristic p [HS93], when R is either a regular local ring of equi-characteristic 0 or a regular affine ring of equi-characteristic 0 [Lyu93], when R is an unramified regular local ring of mixed characteristic [Lyu00], when R is a smooth $\mathbb{Z}$-algebra [BBL$^+$13], and when either $\dim(R)$ or j is sufficiently small (*cf.* [KS99, BRS00, Hel01]). Examples in [Sin00, Kat02, SS04] show that local cohomology modules may have infinitely many associated primes. However, the following question (*cf.* [HKM09, p. 3194]) remains open:

Question I.1. Let R be a Noetherian commutative ring and I be an ideal. Is $\operatorname{Supp}(H^j_I(R))$ Zariski-closed in $\operatorname{Spec}(R)$ for each integer j?

Note that $\operatorname{Supp}(H^j_I(R))$ being Zariski-closed is equivalent to having finitely many *minimal* associated primes. Hence Question I.1 concerns with a finiteness property of local cohomology modules. [HKM09, p. 3195] states that "Clearly, this question is of central importance in the study of cohomological dimension and understanding the local–global properties of local cohomology." Some positive answers to Question I.1 are known: when $j = 2$ and $H^t_I(R) = 0$ for all $t > 2$ ([HKM09, Theorem 1.2]) and when $R = S/(f)$ where S is a Noetherian regular ring of prime characteristic p [HN17, KZ17].

One of the main results of these chapters is the following:

Theorem A (Theorem III.11)**.** Let S be a Noetherian regular ring of prime characteristic p and f_1, f_2 be a regular sequence in S. Set $R = S/(f_1, f_2)$. Then $\operatorname{Supp}(H^j_I(R))$ is Zariski-closed for each integer j and each ideal I.

Another theorem proven here provides a framework to study $\operatorname{Supp}(H^k_I(R/(\underline{f})))$ via investigating $H^i(K^\bullet(\underline{f}; H^j_I(R))$:

Theorem B (Theorem II.4)**.** Let R be a Noetherian ring, $I = (g_1, \ldots, g_t)$ be an ideal, and $f_1, \ldots, f_c$ be a sequence of elements in R. Assume that

(1) The entries of the E_∞ page of the spectral sequence

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H^j_I(R)) \Rightarrow H^{i+j}(T^\bullet),$$

where $T^\bullet$ denotes the relevant total complex, are Zariski-closed, and that

(2) $f_1, \ldots, f_c$ form a regular sequence in R.

Then $\operatorname{Supp}(H^k_I(R/(f_1, \ldots, f_c)))$ is Zariski-closed for each integer k.

Chapter IV, Chapter V, Chapter VI, and Chapter VII primarily concern two topics, both closely related to DG algebras and universal resolutions. DG algebras are invaluable objects for studying homological properties of rings and modules, as, for one, they can be used to describe any number of chain complexes either relevant to module resolutions, or module resolutions themselves. This makes them critical in particular to *universal resolutions*, which are described by [BCL$^+$25] as “free resolutions over a ring that are defined in a uniform way for all finitely generated modules.” Often this uniformity is to buy us the property that, by applying simple modifications to these resolutions we can build resolutions of the same objects over other rings, which are themselves also often called universal resolutions (which is of course valid under this description given of them). Of course, then, DG algebras are foundational to an understanding of universal resolutions as well as of extremal conditions on resolutions such as Golodity and formality. Chapter IV will comprise an overview of DG algebras and universal resolutions, especially as they relate to the topics in the following chapters.

Chapter V is largely a survey, concerning the relationship between some relatively common conditions on module resolutions, namely Golodity and formality, and A_∞ algebras.

Though this is relationship is not a wholly untreaded path, we believe that these relationships are worth more explicitly detailing and compiling. This section can be said to be motivated by the exploration of the *Koszul homomorphism* in [BCL+25]. A Koszul homomorphism is a kind of surjective local ring homomorphism $Q \to R$ which begets a universal resolution which has many avatars in literature. For example, it recovers that constructed by Priddy [Pri70] which is closely connected with a number of explicit, doable-by-hand constructions of useful complexes such as the Tate resolution [EFS03] and the Koszul dual [BGG78], which make it computationally feasible to test for various ring and module properties. It also recovers that constructed by Shamash [Sha69] relating to higher homotopies which has been elaborated upon and applied extensively [Eis80, AB00a, AG02, Pol19]. It is also connected to myriad other constructions, properties, and topics [AKM88, DE22, Frö75, Avr98, Sta63, Bur15, Bur18].

While [BCL+25] explicitly studies the relationship between Koszul homomorphisms and A_∞ algebras, we believe that there is an opportunity to explore some of these same ideas through the lens of *Massey products*, which are often used to verify by-hand facts about A_∞ algebras, given that the two are, though not quite identical, closely related. We go over some of the definitions of formality in terms of A_∞ algebras while providing some explanations regarding the bridges between them in the language of Massey products. As evidence for our claim, and perhaps as our most novel contribution, we speak also on the bridges between A_∞ algebras and Golodity, an extremal property of resolutions which in the language of Massey products is almost identical in definition to that of formality. Though this, again, is not necessarily new territory, it is the author's understanding that these definitions have not all been compiled in one place before. Furthermore, we explicitly describe a relatively well-known relationship between Golodity and formality in a way slightly different from descriptions seen elsewhere using this framework. Say R is a local ring, and K^R is a Koszul complex on a minimal generating set of the maximal ideal of R. The following facts are well-known (though we do not know of any sources explicitly providing the second):

Fact I.2 (Corollary of Definition V.3 and Theorem V.18)**.** The DG algebra K^R is formal if and only if there is a quasi-isomorphism $\mathrm{H}\left(\overline{K^R}\right) \to K^R$ of A_∞ algebras.

Fact I.3 (Corollary V.24)**.** R is Golod if and only if $\mathrm{H}\left(\overline{K^R}\right)$ has trivial multiplication and there is a quasi-isomorphism $\mathrm{H}\left(\overline{K^R}\right) \to K^R$ of A_∞ algebras.

From these, it is clear that Golodity implies formality, and indeed, proving the second fact from the first is simple. However, using the language of Massey products, we are able to prove the second without use of the first which has otherwise, though likely out of a lack of need, not been done, in a way that allows for a better understand of their relationship and of this implication between them. We will also discuss a paper which explores limitations to the relationship between Massey products and A_∞ algebras relevant to this sort of approach. Our hope is that this chapter will allow readers interested in using A_∞ algebra techniques specifically to prove facts regarding DG algebras to able to confidently translate A_∞ algebra problems in this field into problems regarding Massey products, allowing both for further means of exploration and for potential computational exploitation.

In Chapter VI and Chapter VII we will discuss the cohomological support variety, a child of the aforementioned universal resolution constructed by Shamash [Sha69], considering here specifically its generalization to DG modules over a certain Koszul complex, first made explicit in [Pol19]. Specifically, we will consider its construction when resolving a quotient of an ideal of monomials of degree at least two over a polynomial ring or of a ideal of monomials of degree at least two over a Q-regular sequence. Recent work [BGP24, BGP25] has considered the realizability of cohomological support varieties in various levels of generality. The cohomological support variety concerns rings R with *minimal regular presentations*, defined in two ways:

- $\widehat{R} = Q/I$ with Q a regular local ring and I in the square of the maximal ideal of Q (the *local case*),
- $R = Q/I$ where Q is a positively graded polynomial algebra over a field and I an ideal generated by homogeneous forms of degree at least 2 (the *graded case*).

In these cases, we let k be the residue field, or respectively base field, of R. Say $I = (\boldsymbol{f}) = (f_1, \ldots, f_n)$ and M is an finitely generated R-module. The set of possible values of $V_R(M)$, the subject of the *realizability problem* for cohomological support varieties, has been classified when R is a complete intersection [Ber07] or Golod [BGP24]. Work in [BGP25] restricts attention to the case $M = R = Q/(\boldsymbol{f})$. They further suppose that $\boldsymbol{f}$ consists of *monomials*, referring to monomials on a regular sequence of Q or simply monomials in Q, respectively. They classify the cohomological support varieties of rings with minimal regular presentations in which I has at most 5 generators:

Theorem I.4 ([BGP25, Theorem 6.14, Theorem 6.16])**.** Let Q/I be a minimal regular presentation. Let $\boldsymbol{f}$ be a minimal generating set of I consisting of $n \leq 5$ monomials. Then $\mathrm{V}_R(R)$ is either a coordinate subspace of $\mathbb{A}_k^5$ or a union of two hyperplanes.

Using a Macaulay2 calculation, they find a cohomological support variety of a monomial ideal which is not a union of linear subspaces which is realizable when $n = 6$ [BGP25, Example 6.17]. In doing so, they employ a procedure which can be used to compute the cohomological support variety of an arbitrary monomial ideal. However, this procedure involves the computation of the homology of a square matrix of dimension equal to the sum of the Betti numbers of the ideal (which, for example, will be 2^n whenever for each generator there is some variable such that the power of that variable is largest at that generator [Ale17]), making it relatively manually intractable and potentially computationally expensive. For certain monomial ideals, including those which are *equigenerated*, that is, with a generating set of monomials sharing a degree (a set whose monomials we say are *equidegree*), we offer a theorem which allows for the construction of a more efficient procedure. This theorem considers a 2^n-dimensional space, but rather than calculating the homology of an arbitrary automorphism, allows us to partition the space into grades and consider an automorphism which is a differential of a chain complex, which allows for the computation of the homologies of smaller matrices:

Theorem C (Corollary VI.33)**.** Every cohomological support variety of a ring with a minimal regular presentation given by an equigenerated monomial ideal with n generators is the set of points in $\mathbb{A}_k^n$ such that a chain complex of vector spaces with total dimension 2^n with entries defined by polynomials in the n variables begetting our affine space has non-trivial homology.

This consequently allows us to theoretically verify their computational discovery and extend it to a second such variety:

Theorem D (cf. [BGP25, Example 6.17])**.** If

$$R = k[x_1, \dots, x_6]/(x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1),$$

then

$$\mathrm{V}_R(R) = \mathscr{V}(a_1a_3a_5 + a_2a_4a_6) \subseteq \mathbb{A}_k^6.$$

Furthermore, if $\mathrm{char}(k) \notin \{2, 5\}$ and

$$R = k[x_1, \dots, x_{10}]/(x_1x_2, x_2x_3, \dots, x_{10}x_1),$$

then

$$\mathrm{V}_R(R) = \mathscr{V}(a_1a_3a_5a_7a_9 + a_2a_4a_6a_8a_{10}) \subseteq \mathbb{A}_k^{10}.$$

It also allows us to write a method in Macaulay2 which computes cohomological support varieties of monomial ideals over $\mathbb{Q}$. This allows us to make two additional statements, albeit verified only with computer assistance. First, we are able to give another example of a previously-unknown realizable variety, given by the edge ideal of a 14-cycle over $\mathbb{Q}$:

Computation E (Computation VII.4)**.** The edge ideal of a 14-cycle over $\mathbb{Q}$ has support variety

$$\mathscr{V}(a_1a_3 \cdots a_{13} + a_2a_4 \cdots a_{14}).$$

The second is a partial generalization of Theorem I.4 to the case $n = 6$, under the condition that we only consider equigenerated monomial ideals and only over $\mathbb{Q}$:

Computation F (Computation VII.5)**.** The cohomological support varieties of rings over $\mathbb{Q}$ with minimal regular presentations given by ideals with minimal generating sets with 6 equidegree monomials are all one of the following up to order:

- a linear subspace,
- a union of two hyperplanes,
- $\mathscr{V}(a_{135} + a_{246})$.

II

COMPLEXES RELEVANT TO SUPPORT OF LOCAL COHOMOLOGY MODULES

In this chapter, we will explore chain complexes relevant to to the calculation of the support of local cohomology modules. As mentioned in the introduction, one of the main results shown over the course of this chapter and the next is the following:

Theorem A (Theorem III.11)**.** Let S be a Noetherian regular ring of prime characteristic p and f_1, f_2 be a regular sequence in S. Set $R = S/(f_1, f_2)$. Then $\operatorname{Supp}(H^j_I(R))$ is Zariski-closed for each integer j and each ideal I.

Our strategy to prove Theorem A is to link the Koszul cohomology groups of $H^j_I(R)$ on a sequence $\underline{f}$ to the local cohomology modules $H^i_I(R/(\underline{f}))$ via a double complex. To wit, let R be a Noetherian ring and $\underline{f} = f_1, \dots, f_c$ be a sequence of elements. Let $I = (g_1, \dots, g_t)$ be an ideal in R. Let $\check{C}^\bullet(\underline{g}; N)$ denote the Čech complex of an R-module N on the sequence $\underline{g}$ and let $K^\bullet(\underline{f}; N)$ denote the Koszul (co)complex of an R-module N on the sequence $\underline{f}$. Let $\mathbf{D}$ denote the double complex whose i-the row is the Čech complex $\check{C}^\bullet(\underline{g}; K^i(\underline{f}; R))$ and whose j-th column is the Koszul (co)complex $K^\bullet(\underline{f}; C^j(\underline{g}; R))$. Then there is a spectral sequence

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H^j_I(R)) \Rightarrow H^{i+j}(T^\bullet)$$

associated with $\mathbf{D}$, where $T^\bullet$ denotes the total complex of $\mathbf{D}$ (*cf.* Section 1 for details). The following theorem provides a framework to study $\operatorname{Supp}(H^k_I(R/(\underline{f})))$ via investigating $H^i(K^\bullet(\underline{f}; H^j_I(R)))$.

Theorem B (Theorem II.4)**.** Let R be a Noetherian ring, $I = (g_1, \dots, g_t)$ be an ideal, and $f_1, \dots, f_c$ be a sequence of elements in R. Let $E_2^{\bullet,\bullet}$ be as above. Assume that

(1) $\mathrm{Supp}(E_\infty^{i,j})$ are Zariski-closed for all integers i, j, and that

(2) $f_1, \dots, f_c$ form a regular sequence in R.

Then $\mathrm{Supp}(H_I^k(R/(f_1, \dots, f_c)))$ is Zariski-closed for each integer k.

This chapter is organized as follows. In Section 1, we introduce and study a double complex which links the Koszul cohomology of $H_I^j(R)$ on a sequence $\underline{f}$ and the local cohomology modules $H_I^j(R/(\underline{f}))$ and prove Theorem B; Section 1 is characteristic-free and does not require R to be regular. In Section 2, we introduce the notion of the (Frobenius) truncation of Čech complexes which is one of the main technical tools used between Chapter II and Chapter III.

1. A Koszul-Čech double complex and related spectral sequences

Let R be a commutative Noetherian ring and $f_1, \dots, f_c$ and $g_1, \dots, g_t$ be two sequences of elements in R. Set $I = (g_1, \dots, g_t)$ to be the ideal generated by $g_1, \dots, g_t$. For each R-module N,

(1) we denote by $K^\bullet(\underline{f}; N)$ the Koszul co-complex of N on the elements $f_1, \dots, f_c$, which is the R-dual of the Koszul complex $K_\bullet(\underline{f}; N)$, and

(2) we denote by $\check{C}^\bullet(\underline{g}; N)$ the Čech complex of N on $g_1, \dots, g_t$:

$$0 \to N \xrightarrow{\delta^0} \oplus_{i=1}^t N_{g_i} \xrightarrow{\delta^1} \oplus_{i_1<i_2} N_{g_{i_1} g_{i_2}} \xrightarrow{\delta^2} \cdots \to N_{g_1 \cdots g_t} \to 0,$$

where δ^i is defined via $\delta^i : N_{g_{j_1} \cdots g_{j_i}} \to N_{g_{\ell_1} \cdots g_{\ell_{i+1}}}$ is defined as

$$\delta^i\Big(\frac{z}{g_{j_1}^n \cdots g_{j_i}^n}\Big) = \begin{cases} (-1)^{s-1} \frac{z}{g_{j_1}^n \cdots g_{j_i}^n} & \text{when } j_1 \cdots j_i = \ell_1 \cdots \hat{\ell}_s \cdots \ell_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

for each $z \in N$. Note that $H^j(\check{C}^\bullet(\underline{g}; N)) \cong H_I^j(N)$.

Definition II.1. The double complex, denoted by $\mathbf{D} := D(K^\bullet(\underline{f}); \check{C}^\bullet(\underline{g}))$ is the double complex whose i-the row is the Čech complex $\check{C}^\bullet(\underline{g}; K^i(\underline{f}; R))$ and whose j-th column is the Koszul (co)complex $K^\bullet(\underline{f}; C^j(\underline{g}; R))$.

We will denote the total complex of $\mathbf{D}$ by $T^\bullet$.

Example II.2 (When $\boldsymbol{t} = \mathbf{2}$)**.** The most relevant case for this work is when $t = 2$ and we would like to spell out the double complex as follows. The Koszul (co)complex $K^\bullet(f_1, f_2; N)$ is the following for each R-module N:

$$0 \to N \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} N^{\oplus 2} \xrightarrow{(-f_2 \quad f_1)} N \to 0$$

The Koszul-Čech double complex in this case is the following:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \to & R & \to & \bigoplus_j R_{g_j} & \to & \bigoplus_{j_1<j_2} R_{g_{j_1}g_{j_2}} & \to & \cdots & \to & R_{g_1\cdots g_t} & \to 0 \\
 & & \uparrow{\scriptstyle (-f_2\ \ f_1)} & & \uparrow{\scriptstyle (-f_2\ \ f_1)} & & \uparrow{\scriptstyle (-f_2\ \ f_1)} & & & & \uparrow{\scriptstyle (-f_2\ \ f_1)} \\
0 & \to & R^{\oplus 2} & \to & (\bigoplus_j R_{g_j})^{\oplus 2} & \to & (\bigoplus_{j_1<j_2} R_{g_{j_1}g_{j_2}})^{\oplus 2} & \to & \cdots & \to & (R_{g_1\cdots g_t})^{\oplus 2} & \to 0 \\
 & & \uparrow{\scriptstyle \binom{f_1}{f_2}} & & \uparrow{\scriptstyle \binom{f_1}{f_2}} & & \uparrow{\scriptstyle \binom{f_1}{f_2}} & & & & \uparrow{\scriptstyle \binom{f_1}{f_2}} \\
0 & \to & R & \to & \bigoplus_j R_{g_j} & \to & \bigoplus_{j_1<j_2} R_{g_{j_1}g_{j_2}} & \to & \cdots & \to & R_{g_1\cdots g_t} & \to 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
 & & 0 & & 0 & & 0 & & & & 0
\end{array} \tag{II.1}$$

Remark II.3. As discussed in [Wei94, §5.1], there are two spectral sequences associated with our complex $D(K^\bullet(\underline{f}); \check{C}^\bullet(\underline{g}))$.

One of the them comes from taking horizontal differentials (in the Čech complexes) first and then vertical differentials (in the resulted Koszul co-complexes). The resulted spectral sequence is:

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H_I^j(R)) \Rightarrow H^{i+j}(T^\bullet)$$

Recall that $T^\bullet$ is the total complex of $D(K^\bullet(\underline{f}); \check{C}^\bullet(\underline{g}))$.

The other one comes from doing differentials the other way around (considering vertical differentials and then horizontal differentials):

$$'E_2^{i,j} := H_I^i(H^j(K^\bullet(\underline{f}; R)) \Rightarrow H^{i+j}(T^\bullet)$$

The following theorem, one of our main technical tools, indicates the connection between $\mathrm{Supp}(E_\infty^{i,j})$ and $\mathrm{Supp}(H_I^k(R/(f_1, \ldots, f_s)))$ when $f_1, \ldots, f_c$ form a regular sequence in R.

Theorem II.4. Assume that

(1) $\mathrm{Supp}(E_\infty^{i,j})$ are Zariski-closed for all integers i, j, and that
(2) $f_1, \ldots, f_c$ form a regular sequence in R.

Then $\mathrm{Supp}(H_I^k(R/(f_1, \ldots, f_c)))$ is Zariski-closed for each integer k.

Proof. The convergence

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H_I^j(R)) \Rightarrow H^{i+j}(T^\bullet)$$

amounts to a filtration of $H^k(T^\bullet)$ for each k:

$$0 \subseteq F^k H^k(T^\bullet) \subseteq F^{k-1} H^k(T^\bullet) \subseteq \cdots \subseteq F^1 H^k(T^\bullet) \subseteq F^0 H^k(T^\bullet) = H^k(T^\bullet)$$

such that $F^i H^k(T^\bullet)/F^{i+1} H^k(T^\bullet) \cong E_\infty^{i,n-i}$ (with $F^k H^k(T^\bullet) \cong E_\infty^{k,0}$).

Since $E_\infty^{i,j}$ is Zariski closed for all integers i, j, the Zariski-closedness of $\mathrm{Supp}(H^k(T^\bullet))$ follows from the filtration of $H^k(T^\bullet)$.

The assumption that $f_1, \ldots, f_s$ form a regular sequence in R implies that $'E_2^{\bullet,\bullet}$ has only one nonzero row in which the entries are $H_I^i(R/(f_1, \ldots, f_c))$. Consequently, this spectral sequence collapses, providing for each k the isomorphism

$$H^k_I(R/(f_1,\dots,f_c)) \cong H^k(T^\bullet)$$

which shows that $\operatorname{Supp}(H^k_I(R/(f_1,\dots,f_c))$ is Zariski-closed. $\square$

In Section 3, we will prove that $\operatorname{Supp}(E^{i,j}_\infty)$ are Zariski-closed for all integers i,j when R is *regular* of prime characteristic p and $E^{i,j}_\infty$ are associated with the double complex (II.1). One of our technical tools is to truncate the Čech complex.

2. Truncated Čech complexes

In this section we introduce (Frobenius) truncated Čech complexes, one of the main technical tools needed in this work.

Let R be a Noetherian commutative ring of prime characteristic $p>0$ and let $g\in R$ be an element in R. We will use $R\cdot\frac{1}{g^{p^e}}$ denote the cyclic R-submodule of R_f generated by $\frac{1}{g^{p^e}}$, and we will call $R\cdot\frac{1}{g^{p^e}}$ the e-th (Frobenius) truncation of R_g. (Our convention is to consider $R\cdot\frac{1}{g}$ as the 0-th Frobenius truncation of R_g.)

Note that $R\cdot\frac{1}{g^{p^e}}$ is a finitely generated R-module; this finiteness plays a crucial role in this work.

Remark II.5. Let $g_1,\dots,g_t$ be elements in R. Recall that $\check{C}^\bullet(\underline{g};R)$, the Čech complex of R on $g_1,\dots,g_t$, is constructed as follows:

$$0\to R\to\bigoplus_{j=1}^{t}R_{g_j}\to\cdots\to\bigoplus_{j_1<\cdots<j_i}R_{g_{j_1}\cdots g_{j_i}}\xrightarrow{\delta^i}\bigoplus_{j_1<\cdots<j_{i+1}}R_{g_{j_1}\cdots g_{j_{i+1}}}\to\cdots\to R_{g_1\cdots g_t}\to 0$$

where δ^i is defined via $\delta^i: R_{g_{j_1}\cdots g_{j_i}}\to R_{g_{\ell_1}\cdots g_{\ell_{i+1}}}$ is defined as

$$\delta^i\left(\frac{r}{g^n_{j_1}\cdots g^n_{j_i}}\right)=\begin{cases}(-1)^{s-1}\frac{r}{g^n_{j_1}\cdots g^n_{j_i}} & \text{when } j_1\cdots j_i=\ell_1\cdots\hat{\ell}_s\cdots\ell_{i+1}\\ 0 & \text{otherwise}\end{cases} \tag{II.2}$$

Then it is clear that the image of the restriction of δ^i on $R \cdot \frac{1}{g_{j_1}^{p^e} \cdots g_{j_i}^{p^e}}$ is contained in $R \cdot \frac{1}{g_{\ell_1}^{p^e} \cdots g_{\ell_{i+1}}^{p^e}}$. Consequently, if one replaces each module in the Čech complex $C^\bullet(\underline{g}; R)$ by its e-th truncation, then one will get a complex

$$0 \to R \to \bigoplus_{j=1}^{t} R \cdot \frac{1}{g_j^{p^e}} \to \bigoplus_{j_1<j_2} R \cdot \frac{1}{g_{j_1}^{p^e} g_{j_2}^{p^e}} \to \cdots \tag{II.3}$$

Definition II.6. The complex (II.3) is called the e-th truncation of the Čech complex $\check{C}^\bullet(\underline{g}; R)$ and will be denoted by $\check{C}^\bullet(\underline{g}; R)_e$ or $\check{C}^\bullet_e$ when the elements $g_1, \ldots, g_t$ are clear from the context. The i-th term in $\check{C}^\bullet(\underline{g}; R)_e$ will be denoted by $\check{C}^i(\underline{g}; R)_e$ and the i-th differential in $\check{C}^\bullet(\underline{g}; R)_e$ will be denoted by δ^i_e.

For each element $\eta \in \ker(\delta^i)$ (respectively $\eta \in \ker(\delta^i_e)$), its image in $H^i(\check{C}^\bullet(\underline{g}; R))$ (respectively $H^i(\check{C}^\bullet(\underline{g}; R)_e)$) will be denoted by $[\eta]$.

Let R be a Noetherian ring of prime characteristic p. Let $R^{(e)}$ be the additive group of R regarded as an R-bimodule with the usual left R-action and with the right R-action defined by $r'r = r^{p^e} r'$ for all $r \in R$ and $r' \in R^{(e)}$. The e-th Peskine-Szipro functor $\mathbf{F}^e$ is defined via

$$\mathbf{F}^e(M) = R^{(e)} \otimes_R M \quad \mathbf{F}^e(M \xrightarrow{\phi} N) = R^{(e)} \otimes_R M \xrightarrow{1 \otimes \phi} R^{(e)} \otimes_R N.$$

When $e = 1$, we will denote $\mathbf{F}^1$ by $\mathbf{F}$.

Note that, when R is regular, $R^{(e)}$ is a faithfully flat R-module and hence $\mathbf{F}^e$ is an exact functor for each $e \geq 1$ ([Kun69]).

Proposition II.7. Let R be a Noetherian regular ring of prime characteristic $p > 0$ and let $\mathbf{F}$ denote the Peskine-Szpiro functor. Then

(1) $\mathbf{F}(R \cdot \frac{1}{g}) \cong R \cdot \frac{1}{g^p}$ for every $g \in R$.

(2) $\mathbf{F}(\check{C}^\bullet(\underline{g}; R)_e) \cong \check{C}^\bullet(\underline{g}; R)_{e+1}$ for all sequences of elements $\underline{g} = g_1, \ldots, g_t$.

Proof. Note that $\mathbf{F}$ is an exact functor since R is regular.

To prove the first part, it suffices to note that the R linear map

$$\theta : \mathbf{F}(R \cdot \frac{1}{g}) = R^{(1)} \otimes_R R \cdot \frac{1}{g} \xrightarrow{r' \otimes \frac{r}{g} \mapsto \frac{r' r^p}{g^p}} R \cdot \frac{1}{g^p}$$

admits an inverse

$$R \cdot \frac{1}{g^p} \xrightarrow{\frac{r}{g^p} \mapsto r \otimes \frac{1}{g}} R^{(1)} \otimes_R R \cdot \frac{1}{g} = \mathbf{F}(R \cdot \frac{1}{g}).$$

The second part follows from the following commutative diagram

$$\begin{array}{ccc} \mathbf{F}(R \cdot \frac{1}{g_{j_1}^{p^e} \cdots g_{j_i}^{p^e}}) & \longrightarrow & \mathbf{F}(R \cdot \frac{1}{g_{\ell_1}^{p^e} \cdots g_{\ell_{i+1}}^{p^e}}) \\ \downarrow & & \downarrow \\ R \cdot \frac{1}{g_{j_1}^{p^{e+1}} \cdots g_{j_i}^{p^{e+1}}} & \longrightarrow & R \cdot \frac{1}{g_{\ell_1}^{p^{e+1}} \cdots g_{\ell_{i+1}}^{p^{e+1}}} \end{array}$$

where the horizontal maps are induced by the i-th differential (II.2) in the Čech complex and the vertical maps are the isomorphisms in the first part applied to the cases when $g = g_{j_1}^{p^e} \cdots g_{j_i}^{p^e}$ and when $g = g_{\ell_1}^{p^e} \cdots g_{\ell_{i+1}}^{p^e}$, respectively. □

For the rest of this chapter and the next, we will denote by θ the isomorphisms

$$\mathbf{F}^e(\check{C}^j(\underline{g}; R)) \xrightarrow{\sim} \check{C}^j(\underline{g}; R), \quad \mathbf{F}(\check{C}^j(\underline{g}; R)_e) \xrightarrow{\sim} \check{C}^j(\underline{g}; R)_{e+1}$$
$$\text{and} \quad \mathbf{F}^e(\check{C}^j(\underline{g}; R)_0) \xrightarrow{\sim} \check{C}^j(\underline{g}; R)_e.$$

The natural inclusion $R \cdot \frac{1}{g_{j_1}^{p^e} \cdots g_{j_i}^{p^e}} \to R \cdot \frac{1}{g_{j_1}^{p^{e+1}} \cdots g_{j_i}^{p^{e+1}}}$ induces a chain map between the truncated Čech complexes: $\check{C}^\bullet(\underline{g}; R)_e \to \check{C}^\bullet(\underline{g}; R)_{e+1}$ and hence induces an R-module homomorphism $H^i(\check{C}^\bullet(\underline{g}; R)_e) \to H^i(\check{C}^\bullet(\underline{g}; R)_{e+1})$. This produces a directed system:

$$H^i(\check{C}^\bullet(\underline{g}; R)_0) \to H^i(\check{C}^\bullet(\underline{g}; R)_1) \to \cdots \to H^i(\check{C}^\bullet(\underline{g}; R)_e) \to \cdots$$

whose direct limit is isomorphic to $H^i_I(R)$.

Each element in $H^i_I(R)$ can be represented by a cohomological class of the form $[\cdots, \frac{r}{g_{j_1}^n \cdots g_{j_i}^n}, \cdots]$. Let $H^j_I(R)_e$ be the R-submodule of $H^j_I(R)$ generated by classes $[\cdots, \frac{r}{g_{j_1}^n \cdots g_{j_i}^n}, \cdots]$ with $n \leq p^e$. Then $H^j_I(R)_e$ is precisely the image of $H^i(\check{C}^\bullet(\underline{g}; R)_e)$ in

$H^j_I(R)$; consequently $H^i_I(R)_e$ is finitely generated. Furthermore, one can check that

$$H^i_I(R)_e \cong \frac{\ker(\delta^i_e)}{\mathrm{image}(\delta^{i-1}) \cap \ker(\delta^i_e)} \text{ and } \mathbf{F}(H^i_I(R)_e) \cong H^i_I(R)_{e+1}. \tag{II.4}$$

For the rest of this chapter and the next, whenever it is clear from the context, we will write $\check{C}^\bullet(\underline{g})$, or even $\check{C}^\bullet$, instead of $\check{C}^\bullet(\underline{g};R)$.

One can replace the Čech complex with its (Frobenius) truncations in Definition II.1 to form the double complex

$$\mathbf{D}_e := D(K^\bullet(\underline{f^{p^e}}); \check{C}^\bullet(\underline{g})_e)$$

for each integer $e \geq 0$:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \to & R & \to & \bigoplus_j R\cdot\frac{1}{g_j^{p^e}} & \to & \bigoplus_{j_1<j_2} R\cdot\frac{1}{g_{j_1}^{p^e}g_{j_2}^{p^e}} & \to & \cdots & \to & R\cdot\frac{1}{g_1^{p^e}\cdots g_s^{p^e}} & \to & 0 \\
 & & \uparrow{\scriptstyle(-f_2^{p^e}\ f_1^{p^e})} & & \uparrow{\scriptstyle(-f_2^{p^e}\ f_1^{p^e})} & & \uparrow{\scriptstyle(-f_2^{p^e}\ f_1^{p^e})} & & & & \uparrow{\scriptstyle(-f_2^{p^e}\ f_1^{p^e})} \\
0 & \to & R^{\oplus 2} & \to & (\bigoplus_j R\cdot\frac{1}{g_j^{p^e}})^{\oplus 2} & \to & (\bigoplus_{j_1<j_2} R\cdot\frac{1}{g_{j_1}^{p^e}g_{j_2}^{p^e}})^{\oplus 2} & \to & \cdots & \to & (R\cdot\frac{1}{g_1^{p^e}\cdots g_s^{p^e}})^{\oplus 2} & \to & 0 \\
 & & \uparrow{\scriptstyle\binom{f_1^{p^e}}{f_2^{p^e}}} & & \uparrow{\scriptstyle\binom{f_1^{p^e}}{f_2^{p^e}}} & & \uparrow{\scriptstyle\binom{f_1^{p^e}}{f_2^{p^e}}} & & & & \uparrow{\scriptstyle\binom{f_1^{p^e}}{f_2^{p^e}}} \\
0 & \to & R & \to & \bigoplus_j R\cdot\frac{1}{g_j^{p^e}} & \to & \bigoplus_{j_1<j_2} R\cdot\frac{1}{g_{j_1}^{p^e}g_{j_2}^{p^e}} & \to & \cdots & \to & R\cdot\frac{1}{g_1^{p^e}\cdots g_s^{p^e}} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
 & & 0 & & 0 & & 0 & & & & 0
\end{array} \tag{II.5}$$

A priori, one can form the double complex $D(K^\bullet(\underline{f^{p^e}}); \check{C}^\bullet(\underline{g})_{e'})$ for two different integers e and e'. Since this is not needed in this work, we opt not to explore it here.

We will denote the total complex of (II.5) by $T^\bullet_e$. When taking the horizontal differentials (those in the truncated Čech complexes) and then the vertical differentials in (II.5), one obtains a spectral sequence:

$$E^{i,j}_{2,e} := H^i(K^\bullet(\underline{f}; H^j(\check{C}_e)) \Rightarrow H^{i+j}(T^\bullet_e) \tag{II.6}$$

We will denote the differentials in (II.6) by

$$\varphi_{2,e}^{i,j} : E_{2,e}^{i,j} \to E_{2,e}^{i+2,j-1}.$$

Since $\mathbf{F}$ is an exact functor, one can check $\mathbf{F}^e(K^\bullet(\underline{f}; R)) \cong K^\bullet(\underline{f}^{p^p}; R)$ for any sequence $\underline{f}$ of elements in R. On the other hand, according to Proposition II.7 that $\mathbf{F}^e(\check{C}^\bullet(\underline{g})_0) \cong \check{C}^\bullet(\underline{g})_e$ for any sequence $\underline{g}$ of elements in R. Consequently, the double complex $\mathbf{D}_e$ can be obtained by applying $\mathbf{F}^e$ to $\mathbf{D}_0$.

According to Theorem II.4, it suffices to analyze the double complex $\mathbf{D}$. One of our motivations to introduce the double complexes $\mathbf{D}_e$ is that a great deal of information of $\mathbf{D}$ is already encoded in $\mathbf{D}_0$ in which every module is finitely generated. As shown in the sequel, one can link $\mathbf{D}_0$ with $\mathbf{D}$ using the Peskine-Szpiro functor $\mathbf{F}$. This link is rather intricate since $\mathbf{D}_0$ is directly linked with $\mathbf{D}_e$ via $\mathbf{F}^e$ (the differentials in the Koszul (co)complex in $\mathbf{D}_e$ come from the elements $f_1^{p^e}, f_2^{p^e}$, not f_1, f_2).

III

KOSZUL COHOMOLOGY AND SUPPORT OF LOCAL COHOMOLOGY MODULES

1. Koszul cohomology of F-finite F-modules

This chapter is organized as follows. In Section 1 and Section 2, we prove that $H^i(K^\bullet(f_1, f_2; \mathscr{M}))$ has Zariski-closed support when f_1, f_2 form a regular sequence in regular ring R of prime characteristic p and $\mathscr{M}$ is an F-finite F-module. In Section 3, we complete the proof of Theorem A.

Let R be a Noetherian *regular* ring of prime characteristic $p > 0$. In this section, we will investigate $E_2^{i,j}$ in the $E_2^{\bullet,\bullet}$-page coming from the double complex $\mathbf{D}$ has Zariski-closed support; that is the Koszul cohomology $H^i(K^\bullet(\underline{f}; H^j_I(R)))$. Instead of local cohomology modules $H^j_I(R)$, we will consider all F-finite F-modules. To this end, we begin by recalling the definition and basic facts of F-modules (*cf.* [Lyu97]).

(1) An R-module $\mathscr{M}$ is an F*-module* if there is an R-module isomorphism

$$\theta : \mathscr{M} \to \mathbf{F}(\mathscr{M}) = R^{(1)} \otimes_R \mathscr{M}$$

called the structure isomorphism.

(2) If $(\mathscr{M}, \theta_{\mathscr{M}})$ and $(\mathscr{N}, \theta_{\mathscr{N}})$ are F-modules, then an F*-module morphism* from $(\mathscr{M}, \theta_{\mathscr{M}})$ to $(\mathscr{N}, \theta_{\mathscr{N}})$ consists of the the following commutative diagram:

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\varphi} & \mathscr{N} \\ \downarrow \theta_{\mathscr{M}} & & \downarrow \theta_{\mathscr{N}} \\ R^{(1)} \otimes_R \mathscr{M} & \xrightarrow{\mathbf{1} \otimes \varphi} & R^{(1)} \otimes_R \mathscr{N} \end{array}$$

We will simply write this F-module morphism as $\varphi : \mathscr{M} \to \mathscr{N}$ whenever the context is clear.

(3) A *generating morphism* of an F-module is an R-module homomorphism $\beta : M \to \mathbf{F}(M)$, where M is an R-module, such that $\mathscr{M}$ is the direct limit of the top row of the commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\beta} & \mathbf{F}(M) & \xrightarrow{\mathbf{F}(\beta)} & \mathbf{F}^2(M) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathbf{F}(\beta)} & & \downarrow & & \\ \mathbf{F}(M) & \xrightarrow{\mathbf{F}(\beta)} & \mathbf{F}^2(M) & \xrightarrow{\mathbf{F}^2(\beta)} & \mathbf{F}^3(M) & \longrightarrow & \cdots \end{array}$$

and the structure isomorphism $\theta : \mathscr{M} \to \mathbf{F}(\mathscr{M})$ is induced by the vertical morphism in the diagram.

(4) An F-module $\mathscr{M}$ is *F-finite* if it admits a generating morphism $\beta : M \to \mathbf{F}(M)$ where M is a finitely generated R-module.

(5) Each F-finite F-module $\mathscr{M}$ admits an injective generating morphism $\beta : M \hookrightarrow \mathbf{F}(M)$ where M is a finitely generated R-module; (M, β) is called a root of $\mathscr{M}$.

(6) For each $f \in R$, the localization R_f is an F-finite F-module.

(7) Given elements $g_1, \dots, g_s \in R$, the Čech complex $\check{C}^\bullet(\underline{g}; R)$ is a complex in the category of F-finite F-modules; that is, each module $\check{C}^j$ is an F-finite F-module and the differentials δ^j in this complex are F-module morphisms.

(8) $\ker(\delta^j)$ and $\mathrm{image}(\delta^j)$ are F-finite F-modules and consequently $H^j_I(R)$ is an F-finite F-module for each integer j and each ideal I in R.

Let $\mathscr{M}$ be an F-finite F-module and $\beta : M \hookrightarrow \mathbf{F}(M)$ is a root. Let $R^b \xrightarrow{A} R^a \to M \to 0$ be a presentation of M where A is an $a \times b$ matrix whose entries are elements of R. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} R^b & \xrightarrow{A} & R^a & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle \beta} & & \\ R^b & \xrightarrow{A^{[p]}} & R^a & \longrightarrow & F(M) & \longrightarrow & 0 \end{array}$$

where $A^{[p]}$ denotes the matrix whose entries are the p-th powers of the corresponding entries in A and U is an $a \times a$ matrix with entries in R. To ease notation, we will denote this diagram by

$$\operatorname{coker}(A) \xrightarrow{U} \operatorname{coker}(A^{[p]}).$$

Let $f_1, \ldots, f_c$ be a sequence of elements in R and let $H^i(\underline{f}; -)$ denote the i-th Koszul cohomology functor. That is,

$$H^c(\underline{f}; N) \cong N/(\underline{f}N) \quad \text{and} \quad H^0(\underline{f}; N) \cong \bigcap_{j=1}^{t} \ker(N \xrightarrow{f_j} N)$$

for each R-module N.

Theorem III.1. For each F-finite F-module $\mathscr{M}$, we have that $\operatorname{Supp}(H^c(\underline{f}; \mathscr{M}))$ and $\operatorname{Supp}(H^0(\underline{f}; \mathscr{M}))$ are Zariski-closed, where $\underline{f} = \{f_1, \ldots, f_c\}$ is an arbitrary sequence of elements in R.

Before we proceed to the proof, we remark that the special case of Theorem III.1 when $c = 1$ and $\mathscr{M} = H^j_I(R)$ recovers [HN17, Theorem 1.1] and [KZ17, Theorem 7.1(c)].

Proof of Theorem III.1. To treat the 0-th Koszul cohomology, we consider the following diagram:

$$\begin{array}{ccccccc}
\operatorname{coker}(A) & \xrightarrow{U} & \operatorname{coker}(A^{[p]}) & \xrightarrow{U^{[p]}} & \operatorname{coker}(A^{[p^2]}) & \xrightarrow{U^{[p^2]}} & \cdots \\
\Big\downarrow \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} & & \Big\downarrow \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} & & \Big\downarrow \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} & & \\
\operatorname{coker}(A)^{\oplus c} & \xrightarrow{U^{\oplus c}} & \operatorname{coker}(A^{[p]})^{\oplus c} & \xrightarrow{(U^{[p]})^{\oplus c}} & \operatorname{coker}(A^{[p^2]})^{\oplus c} & \xrightarrow{(U^{[p^2]})^{\oplus c}} & \cdots
\end{array} \tag{III.1}$$

Each square in this commutative diagram

$$\begin{array}{ccc} \operatorname{coker}(A^{[p^e]}) & \xrightarrow{U^{[p^e]}} & \operatorname{coker}(A^{[p^{e+1}]}) \\ \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} \Big\downarrow & & \begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix} \Big\downarrow \\ \operatorname{coker}(A^{[p^e]})^{\oplus c} & \xrightarrow{U^{[p^e]}} & \operatorname{coker}(A^{[p^{e+1}]})^{\oplus c} \end{array}$$

commutes since $U^{[p^e]} f_j = f_j U^{[p^e]}$ for each f_j. Therefore (III.1) is a commutative diagram. One can check that the direct limit of (III.1) is

$$\mathscr{M} \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix}} \mathscr{M}^{\oplus c}.$$

It follows from the proof of [KZ17, Theorem 7.1] that

$$\operatorname{Supp}(\ker(\mathscr{M} \xrightarrow{f_j} \mathscr{M})) = \operatorname{Supp}(\frac{(\ker(U^{[p^j]} \cdots U)) :_{R^a} f_j}{\ker(U^{[p^j]} \cdots U)}),\ j \gg 0.$$

Consequently

$$\operatorname{Supp}(H^0(\underline{f}; \mathscr{M})) = \operatorname{Supp}(\frac{(\ker(U^{[p^j]} \cdots U)) :_{R^a} (f_1, \ldots, f_c)}{\ker(U^{[p^j]} \cdots U)}),\ j \gg 0$$

which is Zariski-closed.

To handle the c-th Koszul cohomology, we consider the following diagram:

$$\begin{array}{ccccccc} \operatorname{coker}(A)^{\oplus c} & \xrightarrow{U^{\oplus c}} & \operatorname{coker}(A^{[p]})^{\oplus c} & \xrightarrow{(U^{[p]})^{\oplus c}} & \operatorname{coker}(A^{[p^2]})^{\oplus c} & \xrightarrow{(U^{[p^2]})^{\oplus c}} & \cdots \\ {\scriptstyle (f_1, \cdots, f_c)} \Big\downarrow & & {\scriptstyle (f_1, \cdots, f_c)} \Big\downarrow & & {\scriptstyle (f_1, \cdots, f_c)} \Big\downarrow & & \\ \operatorname{coker}(A) & \xrightarrow{U} & \operatorname{coker}(A^{[p]}) & \xrightarrow{U^{[p]}} & \operatorname{coker}(A^{[p^2]}) & \xrightarrow{U^{[p^2]}} & \cdot \end{array} \tag{III.2}$$

Each square in this commutative diagram

$$\begin{array}{ccc} \operatorname{coker}(A^{[p^e]})^{\oplus c} & \xrightarrow{(U^{[p^e]})^{\oplus c}} & \operatorname{coker}(A^{[p^{e+1}]})^{\oplus c} \\ \downarrow{\scriptstyle (f_1,\cdots,f_c)} & & \downarrow{\scriptstyle (f_1,\cdots,f_c)} \\ \operatorname{coker}(A^{[p^e]}) & \xrightarrow{U^{[p^e]}} & \operatorname{coker}(A^{[p^{e+1}]}) \end{array}$$

commutes since $U^{[p^e]}f_j = f_jU^{[p^e]}$ for each f_j. Therefore (III.2) is a commutative diagram. One can check that the direct limit of (III.2) is

$$\mathscr{M}^{\oplus c} \xrightarrow{(f_1,\cdots,f_c)} \mathscr{M}.$$

Each element in $\mathscr{M}$ can be represented by an element $z \in \operatorname{coker}(A^{[p^e]})$ for some e. Let $\mathfrak{p}$ be a prime ideal of R. This element becomes 0 in $H^c(\underline{f};\mathscr{M})_{\mathfrak{p}}$ if and only if there is an integer j such that

$$(U^{[p^{e+j}]}\cdots U^{[p^e]})z \in \Big(\operatorname{image}((f_1,\dots,f_c)) + \operatorname{image}(A^{[p^{e+j+1}]})\Big).$$

Therefore,

$$\begin{aligned} &(H^c(\underline{f};\mathscr{M})_{\mathfrak{p}} = 0 \\ \Leftrightarrow &\bigcup_j \Big((\operatorname{image}((f_1,\dots,f_c)) + \operatorname{image}(A^{[p^{e+j+1}]})) :_{R^a} (U^{[p^{e+j}]}\cdots U^{[p^e]})\Big)_{\mathfrak{p}} = R^a_{\mathfrak{p}},\ \forall\, e. \end{aligned}$$

Recall that M is assumed to have a presentation $R^b \xrightarrow{A} R^a \to M \to 0$.

Since

$$\begin{aligned} &\Big((\operatorname{image}((f_1,\dots,f_c)) + \operatorname{image}(A^{[p^{e+j+1}]})) :_{R^a} (U^{[p^{e+j}]}\cdots U^{[p^e]})\Big)^{[p]} \\ &= (\operatorname{image}((f_1^p,\dots,f_c^p)) + \operatorname{image}(A^{[p^{e+j+2}]})) :_{R^a} (U^{[p^{e+j+1}]}\cdots U^{[p^{e+1}]}) \\ &\subseteq (\operatorname{image}((f_1,\dots,f_c)) + \operatorname{image}(A^{[p^{e+j+2}]})) :_{R^a} (U^{[p^{e+j+1}]}\cdots U^{[p^{e+1}]}), \end{aligned}$$

one can check that

$$(H^t(\underline{f};\mathscr{M})_{\mathfrak{p}}=0 \Leftrightarrow \bigcup_j \Big((\mathrm{image}((f_1,\dots,f_c))+\mathrm{image}(A^{[p^{e+j+1}]})):_{R^a}(U^{[p^{e+j}]}\cdots U^{[p^e]})\Big)_{\mathfrak{p}}=R^a_{\mathfrak{p}}$$

if and only if

$$\begin{aligned}&(H^t(\underline{f};\mathscr{M})_{\mathfrak{p}}=0\\ \Leftrightarrow &\bigcup_j \Big((\mathrm{image}((f_1,\dots,f_c))+\mathrm{image}(A^{[p^{j+1}]})):_{R^a}(U^{[p^j]}\cdots U)\Big)_{\mathfrak{p}}=R^a_{\mathfrak{p}}\ \text{(that is when } e=0).\end{aligned}$$

This proves that

$$\mathrm{Supp}(H^t(\underline{f};\mathscr{M}))=\mathrm{Supp}\left(\frac{R^a}{(\mathrm{image}((f_1,\dots,f_c))+\mathrm{image}(A^{[p^{j+1}]})):_{R^a}(U^{[p^j]}\cdots U)}\right)$$

which is clearly Zariski-closed. □

The most relevant case to this article is when $\underline{f}$ is a regular sequence in R. We pose the following question:

Question III.2. Let R be a Noetherian regular ring of primes characteristic p and $\underline{f}$ be a regular sequence in R. Is it true that $\mathrm{Supp}(H^i(K^\bullet(\underline{f};\mathscr{M}))$ is Zariski-closed for each integer i and each F-finite F-module $\mathscr{M}$?

To the best our knowledge, Question III.2 is open as stated. In the next section, we will show that it has an affirmative answer when $\underline{f}=f_1,f_2$.

2. Regular sequences of length 2

In this section we consider the case when $t=2$; that is, when R is an F-finite Noetherian regular ring of prime characteristic, f_1,f_2 form a regular sequence in R and $\mathscr{M}$ is an F-finite F-module. The main goal in this section is to prove the following result:

Theorem III.3. $\mathrm{Supp}(H^1(K^\bullet(f_1,f_2;\mathscr{M})))$ is Zariski-closed for every F-finite F-module $\mathscr{M}$ and arbitrary elements f_1,f_2 in R.

Before we can prove Theorem III.3, we would like to consider a special case of it:

Theorem III.4. Assume that an F-finite F-module $\mathscr{M}$ is (f_1, f_2)-torsion. Then $\mathrm{Supp}(H^1(K^\bullet(f_1, f_2; \mathscr{M})))$ is Zariski-closed.

Proof. It follows from the following long exact sequence of Koszul cohomology

$$0 \leftarrow H^2(K^\bullet(f_1, f_2; \mathscr{M})) \leftarrow H^1(K^\bullet(f_1; \mathscr{M})) \xleftarrow{f_2} H^1(K^\bullet(f_1; \mathscr{M}))$$
$$\leftarrow H^1(K^\bullet(f_1, f_2; \mathscr{M})) \leftarrow H^0(K^\bullet(f_1; \mathscr{M})) \xleftarrow{f_2} H^0(K^\bullet(f_1; \mathscr{M})) \leftarrow H^0(K^\bullet(f_1, f_2; \mathscr{M})) \leftarrow 0.$$

that

$$\begin{aligned} &\mathrm{Supp}(H^1(K^\bullet(f_1, f_2; \mathscr{M}))) \\ = &\mathrm{Supp}(\mathrm{coker}(H^0(K^\bullet(f_1; \mathscr{M})) \xrightarrow{f_2} H^0(K^\bullet(f_1; \mathscr{M})))) \\ &\quad \bigcup \mathrm{Supp}(\ker(H^1(K^\bullet(f_1; \mathscr{M})) \xrightarrow{f_2} H^1(K^\bullet(f_1; \mathscr{M})))) \end{aligned}$$

Note that swapping f_1 and f_2 does not affect $H^1(K^\bullet(f_1, f_2; \mathscr{M}))$; consequently

$$\mathrm{Supp}(\mathrm{coker}(H^0(K^\bullet(f_2; \mathscr{M})) \xrightarrow{f_1} H^0(K^\bullet(f_2; \mathscr{M})))) \subseteq \mathrm{Supp}(H^1(K^\bullet(f_1, f_2; \mathscr{M}))).$$

Hence

$$\begin{aligned} \mathrm{Supp}(H^1(K^\bullet(f_1, f_2; \mathscr{M}))) = &\mathrm{Supp}(\mathrm{coker}(H^0(K^\bullet(f_1; \mathscr{M})) \xrightarrow{f_2} H^0(K^\bullet(f_1; \mathscr{M})))) \\ &\bigcup \mathrm{Supp}(\mathrm{coker}(H^0(K^\bullet(f_2; \mathscr{M})) \xrightarrow{f_1} H^0(K^\bullet(f_2; \mathscr{M})))) \\ &\bigcup \mathrm{Supp}(\ker(H^1(K^\bullet(f_1; \mathscr{M})) \xrightarrow{f_2} H^1(K^\bullet(f_1; \mathscr{M})))). \end{aligned}$$

First we treat $\mathrm{Supp}(\ker(H^1(K^\bullet(f_1; \mathscr{M})) \xrightarrow{f_2} H^1(K^\bullet(f_1; \mathscr{M}))))$. Note that

$$\ker(H^1(K^\bullet(f_1; \mathscr{M})) \xrightarrow{f_2} H^1(K^\bullet(f_1; \mathscr{M}))) \cong \ker(\frac{\mathscr{M}}{f_1\mathscr{M}} \xrightarrow{f_2} \frac{\mathscr{M}}{f_1\mathscr{M}}) \cong \frac{f_1\mathscr{M} :_{\mathscr{M}} f_2}{f_1\mathscr{M}}.$$

Let L denote a root of $\mathscr{M}$; that is, L is finitely generated R-submodule of $\mathscr{M}$ equipped with an injective R-module morphism $\beta : L \to \mathbf{F}(L)$ that generates the F-module $\mathscr{M}$. We will set $L_e := \mathbf{F}^e(L) \subseteq \mathscr{M}$ and view L_e as a submodule of L_{e+1} via the injective R-module morphism $F^e(\beta)$. Note that $\mathscr{M} = \cup_{e\geq 1} L_e$.

Claim 1. $\operatorname{Supp}\left(\frac{f_1\mathscr{M}:_{\mathscr{M}} f_2}{f_1\mathscr{M}}\right) = \bigcup_{e\geq 1} \operatorname{Supp}\left(\frac{f_1\mathscr{M}\cap L_e:_{L_e} f_2}{f_1\mathscr{M}\cap L_e}\right)$.

Assume that $\frac{f_1\mathscr{M}:_{\mathscr{M}} f_2}{f_1\mathscr{M}} = 0$. For each $e \geq 1$ and $z_e \in (f_1\mathscr{M} \cap L_e :_{L_e} f_2)$, it follows that $f_2 z_e \in f_1\mathscr{M} \cap L_e \subseteq f_1\mathscr{M}$ and consequently $z_e \in f_1\mathscr{M} \cap L_e$. This shows that $\frac{f_1\mathscr{M}\cap L_e:_{L_e} f_2}{f_1\mathscr{M}\cap L_e} = 0$ for each e; that is,

$$\operatorname{Supp}\left(\frac{f_1\mathscr{M} :_{\mathscr{M}} f_2}{f_1\mathscr{M}}\right) \supseteq \bigcup_{e\geq 1} \operatorname{Supp}\left(\frac{f_1\mathscr{M} \cap L_e :_{L_e} f_2}{f_1\mathscr{M} \cap L_e}\right).$$

On the other hand, assume that $\frac{f_1\mathscr{M}\cap L_e:_{L_e} f_2}{f_1\mathscr{M}\cap L_e} = 0$ for each e. For each $z \in f_1\mathscr{M} :_{\mathscr{M}} f_2 \subseteq \mathscr{M}$, there is an e such that $z \in L_e$. Consequently $f_2 z \in f_1\mathscr{M} \cap L_e$ and hence $z \in f_1\mathscr{M} \cap L_e \subseteq f_1\mathscr{M}$ by the assumption. This shows that $\frac{f_1\mathscr{M}:_{\mathscr{M}} f_2}{f_1\mathscr{M}} = 0$; that is,

$$\operatorname{Supp}\left(\frac{f_1\mathscr{M} :_{\mathscr{M}} f_2}{f_1\mathscr{M}}\right) \subseteq \bigcup_{e\geq 1} \operatorname{Supp}\left(\frac{f_1\mathscr{M} \cap L_e :_{L_e} f_2}{f_1\mathscr{M} \cap L_e}\right).$$

This finishes the proof of our Claim 1.

Claim 2. $\operatorname{Supp}(\frac{f_1\mathscr{M}\cap L:_L f_2}{f_1\mathscr{M}\cap L}) = \bigcup_{e\geq 1} \operatorname{Supp}\left(\frac{f_1\mathscr{M}\cap L_e:_{L_e} f_2}{f_1\mathscr{M}\cap L_e}\right)$.

It suffices to show that if $\frac{f_1\mathscr{M}\cap L:_L f_2}{f_1\mathscr{M}\cap L} = 0$ then $\frac{f_1\mathscr{M}\cap L_e:_{L_e} f_2}{f_1\mathscr{M}\cap L_e} = 0$ for each $e \geq 1$. Applying the functor $\mathbf{F}^e(-)$ to the assumption $\frac{f_1\mathscr{M}\cap L:_L f_2}{f_1\mathscr{M}\cap L} = 0$, one deduces that $\frac{f_1^{p^e}\mathscr{M}\cap L_e:_{L_e} f_2^{p^e}}{f_1^{p^e}\mathscr{M}\cap L_e} = 0$; that is,

$$f_1^{p^e}\mathscr{M} \cap L_e :_{L_e} f_2^{p^e} = f_1^{p^e}\mathscr{M} \cap L_e.$$

Let z_e be an element in $f_1\mathscr{M} \cap L_e :_{L_e} f_2$. Since $\mathscr{M}$ is (f_1, f_2)-torsion, there exists an integer j such that $f_2^{jp^e} z_e = 0$. Since $f_2^{p^e}(f_2^{(j-1)p^e} z_e) = 0 \in f_1^{p^e}\mathscr{M} \cap L_e$, it follows that $f_2^{(j-1)p^e} z_e \in f_1^{p^e}\mathscr{M}\cap L_e$. Repeating this process, one deduces that $z_e \in f_1^{p^e}\mathscr{M}\cap L_e \subseteq f_1\mathscr{M}\cap L_e$. This proves our Claim 2.

Combining these two claims shows that

$$\mathrm{Supp}(\ker(H^1(K^\bullet(f_1;\mathscr{M})) \xrightarrow{f_2} H^1(K^\bullet(f_1;\mathscr{M})))) = \mathrm{Supp}(\frac{f_1\mathscr{M} \cap L :_L f_2}{f_1\mathscr{M} \cap L})$$

which is Zariski closed as L is finitely generated.

It remains to prove that

$$\begin{aligned} &\mathrm{Supp}(\mathrm{coker}(H^0(K^\bullet(f_1;\mathscr{M})) \xrightarrow{f_2} H^0(K^\bullet(f_1;\mathscr{M})))) \\ &\quad \bigcup \mathrm{Supp}(\mathrm{coker}(H^0(K^\bullet(f_2;\mathscr{M})) \xrightarrow{f_1} H^0(K^\bullet(f_2;\mathscr{M}))) \end{aligned}$$

is Zariski closed (which will complete the proof of our lemma).

Note that

$$H^0(K^\bullet(f_1;\mathscr{M})) \cong (0 :_{\mathscr{M}} f_1) \quad \text{and} \quad H^0(K^\bullet(f_2;\mathscr{M})) = (0 :_{\mathscr{M}} f_2)$$

and consequently

$$\begin{aligned} \mathrm{coker}(H^0(K^\bullet(f_1;\mathscr{M})) \xrightarrow{f_2} H^0(K^\bullet(f_1;\mathscr{M}))) &\cong \frac{(0 :_{\mathscr{M}} f_1)}{f_2(0 :_{\mathscr{M}} f_1)} \\ \mathrm{coker}(H^0(K^\bullet(f_2;\mathscr{M})) \xrightarrow{f_1} H^0(K^\bullet(f_2;\mathscr{M}))) &\cong \frac{(0 :_{\mathscr{M}} f_2)}{f_1(0 :_{\mathscr{M}} f_2)} \end{aligned}$$

Since $\mathscr{M} = \cup_{e\geq 0} L_e$, it is straightforward to check that

$$\begin{aligned} \mathrm{Supp}(\frac{(0 :_{\mathscr{M}} f_1)}{f_2(0 :_{\mathscr{M}} f_1)}) &= \bigcup_e \mathrm{Supp}(\frac{(0 :_{L_e} f_1)}{f_2(0 :_{\mathscr{M}} f_1) \cap (0 :_{L_e} f_1)}) \\ \mathrm{Supp}(\frac{(0 :_{\mathscr{M}} f_2)}{f_1(0 :_{\mathscr{M}} f_2)}) &= \bigcup_e \mathrm{Supp}(\frac{(0 :_{L_e} f_2)}{f_1(0 :_{\mathscr{M}} f_2) \cap (0 :_{L_e} f_2)}) \end{aligned} \tag{III.3}$$

Since L is finitely generated and is (f_1, f_2)-torsion, there is an integer e_0 such that

(1) $f_1^{p^{e_0}} L = f_2^{p^{e_0}} L = 0$, and
(2) $f_1(0 :_{\mathscr{M}} f_2) \cap (0 :_L f_2) = f_1(0 :_{L_{e_0}} f_2) \cap (0 :_L f_2)$, and
(3) $f_2(0 :_{\mathscr{M}} f_1) \cap (0 :_L f_1) = f_2(0 :_{L_{e_0}} f_1) \cap (0 :_L f_1)$.

Note that $f_1^{p^{e_0}}L = f_2^{p^{e_0}}L = 0$ implies that

$$f_1^{p^{e_0+e}}L_e = f_2^{p^{e_0+e}}L_e = 0 \tag{III.4}$$

for each integer $e \geq 1$.

Claim 3.

$$\begin{aligned}
&\mathrm{Supp}(\frac{(0:_{\mathscr{M}} f_1)}{f_2(0:_{\mathscr{M}} f_1)}) \cup \mathrm{Supp}(\frac{(0:_{\mathscr{M}} f_2)}{f_1(0:_{\mathscr{M}} f_2)})\\
&= \mathrm{Supp}(\frac{(0:_L f_1)}{f_2(0:_{\mathscr{M}} f_1)\cap(0:_L f_1)}) \cup \mathrm{Supp}(\frac{(0:_{L_{e_0}} f_1)}{f_2(0:_{\mathscr{M}} f_1)\cap(0:_{L_{e_0}} f_1)})\\
&\cup \mathrm{Supp}(\frac{(0:_L f_2)}{f_1(0:_{\mathscr{M}} f_2)\cap(0:_L f_2)}) \cup \mathrm{Supp}(\frac{(0:_{L_{e_0}} f_2)}{f_1(0:_{\mathscr{M}} f_2)\cap(0:_{L_{e_0}} f_2)})
\end{aligned}$$

The inclusion $\supseteq$ follows from (III.3); it remains to show $\subseteq$. To this end, assume that

- $(0:_L f_1) \subseteq f_2(0:_{\mathscr{M}} f_1)$, and
- $(0:_{L_{e_0}} f_1) \subseteq f_2(0:_{\mathscr{M}} f_1)$, and
- $(0:_L f_2) \subseteq f_1(0:_{\mathscr{M}} f_2)$, and
- $(0:_{L_{e_0}} f_2) \subseteq f_1(0:_{\mathscr{M}} f_2)$.

and we need to show $(0:_{\mathscr{M}} f_1) = f_2(0:_{\mathscr{M}} f_1)$ and $(0:_{\mathscr{M}} f_2) = f_1(0:_{\mathscr{M}} f_2)$.

Note it follows from our choice of e_0 that $(0:_L f_1) \subseteq f_2(0:_{L_{e_0}} f_1)$ and $(0:_L f_2) \subseteq f_1(0:_{L_{e_0}} f_2)$.

Given the symmetry between f_1 and f_2, it suffices to show that $(0:_{\mathscr{M}} f_1) = f_2(0:_{\mathscr{M}} f_1)$.

Let $z \in (0:_{\mathscr{M}} f_1)$ be an arbitrary nonzero element. Then $z \in (0:_{L_e} f_1)$ for an integer e since $\mathscr{M} = \cup_e L_e$. It follows from (III.4) that $f_2^{p^{e_0+e}}z = 0$ since $f_2^{p^{e_0+e}}L_e = 0$. That is,

$$\begin{aligned}
z \in (0:_{L_e} f_2^{p^{e_0+e}}) &\subseteq (0:_{L_{e_0+e}} f_2^{p^{e_0+e}})\\
&= \mathbf{F}^{e_0+e}(0:_L f_2) \subseteq \mathbf{F}^{e_0+e}(f_1(0:_{L_{e_0}} f_2)) = f_1^{p^{e_0+e}}(0:_{L_{2e_0+e}} f_2^{e_0+e}))
\end{aligned}$$

Hence, there is an $y \in (0:_{L_{2e_0+e}} f_2^{e_0+e}))$ such that $z = f_1^{p^{e_0+e}}y = f_1^{p^{e_0+e}-1}(f_1y)$. Note that

$$f_1^{p^{e_0+e}}(f_1y) = f_1f_1^{p^{e_0+e}}y = f_1z = 0$$

which implies that

$$f_1 y \in (0 :_{L_{2e_0+e}} f_1^{p^{e_0+e}}) = \mathbf{F}^{e_0+e}((0 :_{L_{e_0}} f_1)) \subseteq \mathbf{F}^{e_0+e}(f_2(0 :_{\mathscr{M}} f_1)) = f_2^{p^{e_0+e}}(0 :_{\mathscr{M}} f_1^{p^{e_0+e}})$$

Thus, there is an $w \in (0 :_{\mathscr{M}} f_1^{p^{e_0+e}})$ such that $f_1 y = f_2^{p^{e_0+e}} w$. Set

$$x = f_1^{p^{e_0+e}-1} f_2^{p^{e_0+e}-1} w.$$

Then

$$f_2 x = f_2 f_2^{p^{e_0+e}-1} f_1^{p^{e_0+e}-1} x = f_1^{p^{e_0+e}-1} f_2^{p^{e_0+e}} w = f_1^{p^{e_0+e}-1} f_1 y = f_1^{p^{e_0+e}} y = z$$

and

$$f_1 x = f_1 f_2^{p^{e_0+e}-1} f_1^{p^{e_0+e}-1} w = f_2^{p^{e_0+e}-1} f_1^{p^{e_0+e}} w = 0$$

since $f_1^{p^{e_0+e}} w = 0$ by the choice of w. This proves that $z = f_2 x$ and $x \in (0 :_{\mathscr{M}} f_1)$; that is, $z \in f_2(0 :_{\mathscr{M}} f_1)$ and hence completes the proof of our Claim 3.

Note that Claim 3 implies $\mathrm{Supp}(\frac{(0:_{\mathscr{M}} f_1)}{f_2(0:_{\mathscr{M}} f_1)}) \cup \mathrm{Supp}(\frac{(0:_{\mathscr{M}} f_2)}{f_1(0:_{\mathscr{M}} f_2)})$ is Zariski closed since both L and L_{e_0} are finitely generated.

Combining our 3 claims completes the proof of our theorem. □

We now return to the general case when $\mathscr{M}$ is an arbitrary F-finite F-module. Let Γ denote $\Gamma_{(f_1,f_2)}(\mathscr{M})$. The short exact sequence

$$0 \to \Gamma \to \mathscr{M} \to \mathscr{M}/\Gamma \to 0$$

induces an exact sequence on Koszul cohomology

$$\begin{aligned} 0 = H^0(K^\bullet(\underline{f}; \mathscr{M}/\Gamma(\mathscr{M}))) &\to H^1(K^\bullet(\underline{f}; \Gamma)) \\ \to H^1(K^\bullet(\underline{f}; \mathscr{M})) &\to H^1(K^\bullet(\underline{f}; \mathscr{M}/\Gamma)) \xrightarrow{\delta} H^2(K^\bullet(\underline{f}; \Gamma)) \end{aligned} \tag{III.5}$$

The connecting morphism δ can be constructed as follows. Each element in $H^1(\underline{f}; \mathscr{M}/\Gamma)$ can be represented by a pair (a, b) with $-f_2 a + f_1 b = 0 \in \mathscr{M}/\Gamma$ and $a, b \in \mathscr{M}/\Gamma$; equivalently, each element in $H^1(\underline{f}; \mathscr{M}/\Gamma)$ can be represented by a pair (a, b) in $\mathscr{M} \oplus \mathscr{M}$ such that

$-f_2a + f_1b \in \Gamma$. Then

$$\delta(a,b) = \overline{-f_2a + f_1b} \in \frac{\Gamma}{(f_1, f_2)\Gamma} \cong H^2(K^\bullet(\underline{f};\Gamma)).$$

Following notation in the proof of Theorem III.4, we denote by L a root of $\mathscr{M}$; that is, L is a finitely generated R-module with an injective R-module morphism $\beta : L \to \mathbf{F}(L)$ that generates $\mathscr{M}$.

Lemma III.5. $\operatorname{Supp}(\ker(\delta)) = \operatorname{Supp}\left(\frac{(f_1\mathscr{M}\cap L:_L f_2)}{(f_1\mathscr{M}\cap L:_L f_2)\cap(\cup_{j\geq 0}((f_1^{j+1}\mathscr{M}\cap L:_L f_1^j)))}\right)$. In particular, it is Zariski closed.

Proof. First we would like to prove that following claim.

Claim. $\operatorname{Supp}(\ker(\delta)) = \operatorname{Supp}\left(\frac{(f_1\mathscr{M}:_\mathscr{M} f_2)}{(f_1\mathscr{M}:_\mathscr{M} f_2)\cap(\cup_j(f_1^{j+1}\mathscr{M}:_\mathscr{M} f_1^j))}\right)$.

To prove our claim, we show that

$$\ker(\delta) = 0 \Leftrightarrow (f_1\mathscr{M} :_\mathscr{M} f_2) = (f_1\mathscr{M} :_\mathscr{M} f_2) \cap (\cup_j(f_1^{j+1}\mathscr{M} :_\mathscr{M} f_1^j)).$$

Each element in $\ker(\delta)$ can be represented by (a,b) with $a,b \in \mathscr{M}$ such that $f_1b - f_2a \in (f_1,f_2)\Gamma$. That is, there are $u,v \in \Gamma$ such that $f_2b - f_1a = f_1u + f_2v$. By replacing a,b with $a+u, b-v$ (which does not change the images of a,b in $\mathscr{M}/\Gamma$), one can assume that $f_2a = f_1b$.

Assume that $\ker(\delta) = 0$. Given each $a \in (f_1\mathscr{M} :_\mathscr{M} f_2)$, there is an element $b \in \mathscr{M}$ such that $f_2a = f_1b$ and hence (a,b) produces an element in $\ker(\delta)$ which is zero by our assumption. Hence there is an element $c \in \mathscr{M}$ such that

$$(f_1c, f_2c) = (a,b) \in (\mathscr{M}/\Gamma)^{\oplus 2};$$

that is, there is an integer j such that $f_1^j(f_1c - a) = 0$ which implies that $a \in (f_1^{j+1}\mathscr{M} :_\mathscr{M} f_1^j)$. This proves that $(f_1\mathscr{M} :_\mathscr{M} f_2) = (f_1\mathscr{M} :_\mathscr{M} f_2) \cap (\cup_j(f_1^{j+1}\mathscr{M} :_\mathscr{M} f_1^j))$.

On the other hand, assume that $(f_1\mathscr{M} :_\mathscr{M} f_2) = (f_1\mathscr{M} :_\mathscr{M} f_2) \cap (\cup_j(f_1^{j+1}\mathscr{M} :_\mathscr{M} f_1^j))$. Let (a,b) be an element in $\ker(\delta)$. According to the discussion above, we can assume that

$f_2a = f_1b$ and hence $a \in (f_1\mathscr{M} :_{\mathscr{M}} f_2)$. It follows from the assumption that there is an integer j such that $f_1^j a = f_1^{j+1}c$. Then

$$f_1^{j+1}(f_2c - b) = f_2f_1^{j+1}a - f_1^{j+1}b = f_2f_1^j a - f_1^{j+1}b = f_1^{j+1}b - f_1^{j+1}b = 0$$

and hence

$$(f_1c, f_2c) = (a, b) \in (\mathscr{M}/\Gamma)^{\oplus 2}$$

which shows that $(a, b) = 0 \in H^1(\underline{f}; \mathscr{M}/\Gamma)$. This finishes the proof of our claim.

It remains to show that

$$\begin{aligned} &\operatorname{Supp}\Big(\frac{(f_1\mathscr{M} :_{\mathscr{M}} f_2)}{(f_1\mathscr{M} :_{\mathscr{M}} f_2) \cap (\cup_j (f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j))}\Big) \\ &\quad = \operatorname{Supp}\Big(\frac{(f_1\mathscr{M} \cap L :_L f_2)}{(f_1\mathscr{M} \cap L :_L f_2) \cap (\cup_{j\geq 0}((f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)))}\Big) \end{aligned}$$

which is equivalent to proving

$$(f_1\mathscr{M} :_{\mathscr{M}} f_2) \subseteq \cup_{j\geq 0}(f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j) \Leftrightarrow (f_1\mathscr{M} \cap L :_L f_2) \subseteq \cup_{j\geq 0}(f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)$$

We begin with the implication $\Rightarrow$. Assume that $(f_1\mathscr{M} :_{\mathscr{M}} f_2) \subseteq \cup_{j\geq 0}(f_1^{j+1}\mathscr{M} :_{\mathscr{M}} f_1^j)$. Let $a \in (f_1\mathscr{M} \cap L :_L f_2)$ be an arbitrary element. Then, as $L \subseteq \mathscr{M}$, there is an integer j and element $c \in \mathscr{M}$ such that $f_1^j a = f_1^{j+1}c$. This shows that $a \in (f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)$ since $f_1^j a = f_1^{j+1}c \in f_1^{j+1}\mathscr{M} \cap L$. This proves the implication $\Rightarrow$.

We now prove the implication $\Leftarrow$. Assume that $(f_1\mathscr{M} \cap L :_L f_2) \subseteq \cup_{j\geq 0}(f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)$. Let $a \in (f_1\mathscr{M} :_{\mathscr{M}} f_2)$ be an arbitrary element. Then $f_2a = f_1b$ for some element $b \in \mathscr{M}$. Since $\mathscr{M} = \cup_{e\geq 0}L_e$, there is an integer e such that $a \in L_e$.

Apply the functor $\mathbf{F}^e(-)$ to $(f_1\mathscr{M} \cap L :_L f_2) \subseteq \cup_{j\geq 0}(f_1^{j+1}\mathscr{M} \cap L :_L f_1^j)$. Let $a \in (f_1\mathscr{M} :_{\mathscr{M}} f_2)$ implies that

$$(f_1^{p^e}\mathscr{M} \cap L_e :_{L_e} f_2^{p^e}) \subseteq \cup_{j\geq 0}(f_1^{(j+1)p^e}\mathscr{M} \cap L_e :_{L_e} f_1^{jp^e}).$$

The equation $f_2 a = f_1 b$ implies that $f_2^{p^e} f_1^{p^e-1} a = f_1^{p^e} f_2^{p^e-1} b$ and hence

$$f_1^{p^e-1} a \in (f_1^{p^e} \mathscr{M} \cap L_e :_{L_e} f_2^{p^e}) \subseteq \cup_{j\geq 0}(f_1^{(j+1)p^e} \mathscr{M} \cap L_e :_{L_e} f_1^{jp^e}).$$

Therefore, there is an integer ℓ and element $c \in \mathscr{M}$ such that

$$f_1^{(\ell+1)p^e-1} a = f_1^{\ell p^e} f_1^{p^e-1} a = f_1^{(\ell+1)p^e} c$$

which implies that

$$a \in (f_1^{(\ell+1)p^e} \mathscr{M} :_{\mathscr{M}} f_1^{(\ell+1)p^e-1}) \subseteq \cup_{j\geq 0}(f_1^{j+1} \mathscr{M} :_{\mathscr{M}} f_1^{j}).$$

This proves the implication $\Leftarrow$ and hence finishes the proof of our lemma. □

Proof of Theorem III.3. It follows from the exact sequence (III.5) that

$$\mathrm{Supp}(H^1(K^\bullet(f_1, f_2; \mathscr{M}))) = \mathrm{Supp}(H^1(\underline{f}; \Gamma)) \cup \mathrm{Supp}(\ker(\delta)).$$

Combining Theorem III.4 and Lemma III.5 completes the proof. □

Combining Theorem III.1 and Theorem III.3, the following result is immediate:

Theorem III.6. Let R be a Noetherian regular ring of prime characteristic p and $f_1, f_2 \in R$ form a regular sequence. Then, for every F-finite F-module, $\mathrm{Supp}(H^i(K^\bullet(f_1, f_2; \mathscr{M}))$ is Zariski-closed for each integer i.

3. The support of $E_\infty^{\bullet;\bullet}$ when $t = 2$

In this section, we prove that the support of $E_\infty^{i,j}$ is Zariski closed for all integers i, j and the main theorem of this article: Theorem III.11. Let R be a Noetherian commutative ring, $I = (g_1, \ldots, g_s)$ be an ideal and $f_1, f_2 \in R$ be a regular sequence. Then the Koszul (co)complex $K^\bullet(\underline{f}; R)$ and the Čech complex $\check{C}^\bullet(\underline{g}; R)$ induce the double complex (II.1) introduced in Section 1. This double complex induces a spectral sequence whose $E_2^{\bullet;\bullet}$-page is as follows:

$$E_2^{i,j} := H^i(K^\bullet(\underline{f}; H_I^j(R)) \Rightarrow H^{i+j}(T^\bullet).$$

Note that when $t = 2$ there is only one (potentially) nontrivial differential on the E_2-page:

$$d_2^{0,j} : E_2^{0,j} \to E_2^{2,j-1}$$

Consequently

$$E_\infty^{1,j} = E_2^{1,j}, \quad E_\infty^{0,j} = E_3^{0,j} = \ker(d_2^{0,j}), \quad E_\infty^{2,j} = E_3^{2,j} = \operatorname{coker}(d_2^{0,j}). \tag{III.6}$$

We have seen in Section 2 that the support of $E_2^{1,j} = H^1(K^\bullet(f_1, f_2; H_I^j(R)))$ is Zariski closed. It remains to show that both $\operatorname{Supp}(\ker(d_2^{0,j}))$ and $\operatorname{Supp}(\operatorname{coker}(d_2^{0,j}))$ are Zariski-closed. To this end, we begin with analyzing the construction of $d_2^{0,j}$.

Remark III.7. We would like to recall the construction of $d_2^{0,j}$; the interested reader is referred to [Wei94, 5.1.2] for more details. In order to cover the double complexes (II.1) and (II.5), we will consider a first quadrant double complex formed by the Koszul co-complex $K^\bullet(\underline{t}; R)$ on two elements t_1, t_2 and a finite complex $C^\bullet$ of R-modules (differentials in $C^\bullet$ will be denoted by $d_h^\bullet$):

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots & \longrightarrow & C^s & \longrightarrow & 0 \\
 & & {\scriptstyle (-t_2 \;\; t_1)}\uparrow & & {\scriptstyle (-t_2 \;\; t_1)}\uparrow & & {\scriptstyle (-t_2 \;\; t_1)}\uparrow & & & & {\scriptstyle (-t_2 \;\; t_1)}\uparrow & & \\
0 & \longrightarrow & (C^0)^{\oplus 2} & \longrightarrow & (C^1)^{\oplus 2} & \longrightarrow & (C^2)^{\oplus 2} & \longrightarrow & \cdots & \longrightarrow & (C^s)^{\oplus 2} & \longrightarrow & 0 \\
 & & {\scriptstyle \binom{t_1}{t_2}}\uparrow & & {\scriptstyle \binom{t_1}{t_2}}\uparrow & & {\scriptstyle \binom{t_1}{t_2}}\uparrow & & & & {\scriptstyle \binom{t_1}{t_2}}\uparrow & & \\
0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots & \longrightarrow & C^s & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array} \tag{III.7}$$

Each element $[\eta] \in H^0(K^\bullet(t_1, t_2; H^j(C^\bullet))$ is an element $[\eta] \in H^j(C^\bullet)$ such that $(t_1[\eta], t_2[\eta]) = (0, 0) \in (H^j(C^\bullet))^{\oplus 2}$; equivalently $[\eta]$ can be represented by element $\eta \in C^j$ such that $d_h^j(\eta) = 0$ and there are elements $(\alpha_1, \alpha_2) \in (C^{j-1})^{\oplus 2}$ such that

$$d_h^{j-1}(\alpha_1) = t_1\eta \quad \text{and} \quad d_h^{j-1}(\alpha_2) = t_2\eta.$$

Consider $-t_2\alpha_1 + t_1\alpha_2 \in C^{j-1}$. Since

$$d_h^{j-1}(-t_2\alpha_1 + t_1\alpha_2) = -t_2 d_h^{j-1}(\alpha_1) + t_1 d_h^{j-1}(\alpha_2) = -t_2t_1\eta + t_1t_2\eta = 0$$

the element $-t_2\alpha_1 + t_1\alpha_2 \in C^{j-1}$ represents an element $[-t_2\alpha_1 + t_1\alpha_2] \in H^{j-1}(C^\bullet)$. Then

$$d_2^{0,j}([\eta]) = \overline{[-t_2\alpha_1 + t_1\alpha_2]} \in E_2^{2,j-1} = H^2(K^\bullet(f_1, f_2; H^{j-1}(C^\bullet))) \cong \frac{H^{j-1}(C^\bullet)}{(t_1, t_2)H^{j-1}(C^\bullet)}.$$

For instance, the edge map in the spectral sequence associated with the double complex (II.5)

$$\varphi_{2,e}^{0,j} : H^0(K^\bullet(f_1^{p^e}, f_2^{p^e}; H^j(\check{C}^\bullet(\underline{g})_e)) \to H^2(K^\bullet(f_1^{p^e}, f_2^{p^e}; H^{j-1}(\check{C}^\bullet(\underline{g})_e))$$

can be described as follows. Each element $[\eta] \in H^0(K^\bullet(f_1^{p^e}, f_2^{p^e}; H^j(\check{C}^\bullet(\underline{g})_e))$ is an element $[\eta] \in H^j(\check{C}^\bullet(\underline{g})_e)$ such that $(f_1^{p^e}[\eta], f_2^{p^e}[\eta]) = (0, 0) \in (H^j(\check{C}^\bullet(\underline{g})_e))^{\oplus 2}$; equivalently $[\eta]$ can be represented by element $\eta \in \check{C}^j(\underline{g})_e$ such that $\delta^j(\eta) = 0$ and there are elements $\alpha_1, \alpha_2 \in \check{C}^{j-1}(\underline{g})_e$ such that

$$\delta^{j-1}(\alpha_1) = f_1^{p^e}\eta \quad \text{and} \quad \delta^{j-1}(\alpha_2) = f_2^{p^e}\eta.$$

Consider $-f_2^{p^e}\alpha_1 + f_1^{p^e}\alpha_2 \in C_e^{j-1}$. Since

$$\delta^{j-1}(-f^{p^e}2\alpha_1 + f_1^{p^e}\alpha_2) = -f_2^{p^e}\delta^{j-1}(\alpha_1) + f_1^{p^e}\delta^{j-1}(\alpha_2) = -f_2^{p^e}f_1^{p^e}\eta + f_1^{p^e}f_2^{p^e}\eta = 0$$

the element $-f_2^{p^e}\alpha_1 + f_1^{p^e}\alpha_2 \in \check{C}^{j-1}(\underline{g})$ represents an element $[-f_2^{p^e}\alpha_1 + f_1^{p^e}\alpha_2] \in H_I^{j-1}(R)$. Then

$$\begin{aligned}\varphi_{2,e}^{0,j}([\eta]) &= \overline{[-f_2^{p^e}\alpha_1 + f_1^{p^e}\alpha_2]} \\ &\in H^2(K^\bullet(f_1^{p^e}, f_2^{p^e}; H^{j-1}(\check{C}_e^\bullet))) \cong \frac{H^{j-1}(\check{C}^\bullet(\underline{g})_e)}{(f_1^{p^e}, f_2^{p^e})H^{j-1}(\check{C}^\bullet(\underline{g})_e)}.\end{aligned} \tag{III.8}$$

To ease notation, for the rest of this section we will denote the Čech complex $\check{C}^\bullet(\underline{g})$ by $\check{C}^\bullet$ and its e-th truncation $\check{C}^\bullet(\underline{g})_e$ by $\check{C}^\bullet_e$.

Recall that the double complex $\mathbf{D}_0$ induces the spectral sequence (II.6):

$$E_{2,0}^{i,j} := H^i(K^\bullet(\underline{f}; H^j(\check{C}^\bullet_0)) \Rightarrow H^{i+j}(T^\bullet_0)$$

with the differentials

$$\varphi_{2,0}^{i,j} : H^0(K^\bullet(\underline{f}; H^j(\check{C}^\bullet_0)) \to H^2(K^\bullet(\underline{f}; H^{j-1}(\check{C}^\bullet_0)).$$

Let $K_0^j \subseteq \ker(\delta_0^j) \subseteq \check{C}_0^j$ be the submodule whose image in $H^j(\check{C}^\bullet)$ is the kernel of $\varphi_2^{0,j}$, where δ_0^j denotes the j-th differential in $\check{C}^\bullet_0$. Note that

(1) K_0^j is a finitely generated R-module since $\check{C}_0^j$ is so;
(2) the image of $H^j(\check{C}^\bullet_0)$ in $H^j_I(R)$ is isomorphic to $\frac{\ker(\delta_0^j)}{\ker(\delta_0^j)\cap \mathrm{image}(\delta^{j-1})}$, where δ^j denotes the j-th differential in $\check{C}^\bullet$; this is contained in (II.4).

First we treat $\mathrm{Supp}(E_\infty^{0,j})$ which is $\mathrm{Supp}(\ker d_2^{0,j})$ (III.6) and we begin with the following lemma.

Lemma III.8. Let R be a Noetherian regular ring of prime characteristic p. Let $\varphi_{2,e}^{0,j}$ be defined as in (III.8). Let K_e^j be the submodule of $\ker(\delta_e^j) \subseteq \check{C}_e^j$ whose image in $H^j(\check{C}^\bullet_e)$ is the kernel of $\varphi_{2,e}^{0,j}$. Let $\theta : \mathbf{F}^e(\check{C}_0^j) \xrightarrow{\sim} \check{C}_e^j$ denote the isomorphism in Proposition II.7. Then

$$\theta(\mathbf{F}^e(K_0^j)) = K_e^j.$$

Proof. This follows from the commutative diagram below and the fact $R^{(e)}$ is a faithfully flat R-module.

$$\begin{array}{ccc}
R^{(e)} \otimes (\check{C}_0^{j-1} \oplus \check{C}_0^{j-1}) & \xrightarrow{\sim} & \check{C}_e^{j-1} \oplus \check{C}_e^{j-1} \\
\Big\downarrow {\scriptstyle \mathbf{1}\otimes(\delta_0^{j-1}\oplus\delta_0^{j-1})} & & \Big\downarrow {\scriptstyle \delta_e^{j-1}\oplus\delta_e^{j-1}} \\
R^{(e)} \otimes (\check{C}_0^{j} \oplus \check{C}_0^{j}) & \xrightarrow{\sim} & \check{C}_e^{j} \oplus \check{C}_e^{j} \\
\Big\uparrow {\scriptstyle \mathbf{1}\otimes\binom{f_1}{f_2}} & & \Big\uparrow {\scriptstyle \mathbf{1}\otimes\binom{f_1^{p^e}}{f_2^{p^e}}} \\
R^{(e)} \otimes \check{C}_0^{j} & \xrightarrow{\sim} & \check{C}_e^{j}
\end{array}$$

□

Theorem III.9. Let R be a Noetherian regular ring of prime characteristic p and let $E_2^{\bullet,\bullet}$ be the E_2-page of the spectral sequence associated with the double complex (II.1). Then $\ker d_2^{0,j} = 0$ if and only if $K_0^j \subseteq \operatorname{image}(\delta^{j-1})$; that is,

$$\operatorname{Supp}(E_\infty^{0,j}) = \operatorname{Supp}(\ker d_2^{0,j}) = \operatorname{Supp}\Big(\frac{K_0^j}{K_0^j \cap \operatorname{image}(\delta^{j-1})}\Big). \tag{III.9}$$

In particular, $\operatorname{Supp}(E_\infty^{0,j}) = \operatorname{Supp}(\ker d_2^{0,j})$ is Zariski-closed.

Proof. The second statement follows from (III.9) since K_0^j is finitely generated.

To complete the proof, it remains to show that $\ker d_2^{0,j} = 0$ if and only if $K_0^j \subseteq \operatorname{image}(\delta^{j-1})$.

Assume that $d_2^{0,j}$ is injective and $[\eta] \in K_0^j$. One needs to show that $[\eta] \in \operatorname{image}(\delta^{j-1})$. Since $[\eta]$ belongs to K_0^j, its image in $H^j(\check{C}_0^\bullet)$ must belong in $\ker(\varphi_{2,0}^{0,j})$. It follows that the image of $[\eta]$ in $H_I^j(R)$ must belong in $\ker(d_2^{0,j})$. Since $d_2^{0,j}$ is injective, the image of $[\eta]$ in $H_I^j(R)$ must be $[0]$, which implies that $[\eta] \in \operatorname{image}(\delta^{j-1})$. This proves the 'if' statement.

Assume that $K_0^j \subseteq \operatorname{image}(\delta^{j-1})$; that is, if $\varphi_{2,0}^{0,j}([\eta]) = [0]$, then $\eta \in \operatorname{image}(\delta^{j-1})$ (equivalently, the image $[\eta]$ of η in $H_I^j(R)$ is zero). Note it follows from Lemma III.8 that

$$K_e^j \cong \mathbf{F}^e(K_0^j) \subseteq \mathbf{F}^e(\operatorname{image}(\delta^{j-1})) \cong \operatorname{image}(\delta^{j-1})$$

where the last isomorphism follows from that fact that δ^{j-1} is a differential in the Čech complex and hence an F-module morphism.

Let $[\tau]$ be an element in $\ker(d_2^{0,j})$, it remains to show that $[\tau] = [0] \in H_I^j(R)$. Since $[\tau] \in \ker(d_2^{0,j})$, there are elements $\tau \in \check{C}^j$ and $\alpha_1, \alpha_2 \in \check{C}^{j-1}$ such that

$$\delta^{j-1}(\alpha_1) = f_1\tau,\ \delta^{j-1}(\alpha_2) = f_2\tau,\ \text{and } d_2^{0,j}([\tau]) = \overline{[-f_2\alpha_+ f_1\alpha_2]} \in (f_1, f_2)H_I^{j-1}(R).$$

Since there are finitely many cohomology classes involved, there exists an integer e such that $\tau \in \check{C}_e^j$, $\alpha_1, \alpha_2 \in \check{C}_e^{j-1}$, and that $d_2^{0,j}([\tau])$ can be represented by an element in $(f_1, f_2)H^{j-1}(\check{C}_e^\bullet)$. We will fix one such e and we consider the double complex (II.5) for this integer e. It follows that

$$\delta^{j-1}(f_1^{p^e-1}\alpha_1) = f_1^{p^e}\tau \text{ and } \delta^{j-1}(f_2^{p^e-1}\alpha_2) = f_2^{p^e}\tau.$$

According to the description of the edge map (III.8) associated with the double complex (II.5):

$$\begin{aligned}
\varphi_{2,e}^{0,j}([\tau]) &= \overline{[-f_2^{p^e} f_1^{p^e-1}\alpha_1 + f_1^{p^e} f_2^{p^e-1}\alpha_2]} \\
&= (f_1^{p^e-1} f_2^{p^e-1})\overline{[-f_2\alpha_+ f_1\alpha_2]} \\
&\in (f_1^{p^e-1} f_2^{p^e-1})(f_1, f_2)H^{j-1}(\check{C}_e^\bullet) \\
&\in (f_1^{p^e}, f_2^{p^e})H^{j-1}(\check{C}_e^\bullet)
\end{aligned}$$

That is $[\tau]$ belongs in K_e^j and consequently $[\tau] \in K_e^j \subseteq \text{image}(\delta^{j-1})$. Thus, the image of $[\tau]$ in $H_I^j(R)$ is zero. This shows that, if $K_0^j \subseteq \text{image}(\delta^{j-1})$, then $d_2^{0,j}$ is injective, which completes the proof. □

Theorem III.10. Let R be a Noetherian regular ring of prime characteristic p and let $E_2^{\bullet,\bullet}$ be the E_2-page of the spectral sequence associated with the double complex (II.1). Let $H \subseteq H_I^{j-1}(R)$ be the submodule generated by elements that can be represented by elements in $\check{C}_0^{j-1}$. Let $L \subseteq H_I^{j-1}(R)$ be the submodule whose image in $H_I^{j-1}(R)/(f_1, f_2)H_I^{j-1}(R)$ is image$(d_2^{0,j})$. Then $d_2^{0,j}$ is surjective if and only if $H \subseteq L$; that is

$$\mathrm{Supp}(E_\infty^{2,j-1}) = \mathrm{Supp}(\operatorname{coker} d_2^{0,j}) = \mathrm{Supp}(\frac{H}{H \cap L}). \tag{III.10}$$

In particular, $\mathrm{Supp}(E_\infty^{2,j-1}) = \mathrm{Supp}(\operatorname{coker} d_2^{0,j})$ is Zariski-closed.

Proof. Since H is finitely generated (II.4), the Zariski-closedness follows from the 'if and only if' statement.

If $d_2^{0,j}$ is surjective, then $L = H_I^{j-1}(R)$ and hence $H \subseteq L$.

Assume that $H \subseteq L$. Then $\mathbf{F}^e(H) \subseteq \mathbf{F}^e(L)$ for each e since $\mathbf{F}$ is an exact functor. Note that $\mathbf{F}^e(H)$ is the submodule of $H_I^{j-1}(R)$ generated by elements that can be represented by elements in $\check{C}_e^{j-1}$ and that $\mathbf{F}^e(L)$ is the submodule of $H_I^{j-1}(R)$ whose image in $H_I^{j-1}(R)/(f_1^{p^e}, f_2^{p^e})H_I^{j-1}(R)$ is image$(\varphi_{2,e}^{0,j})$.

Let $[\eta]$ be an arbitrary element in $H_I^{j-1}(R)/(f_1, f_2)H_I^{j-1}(R)$. Pick an element η_e in $\check{C}_e^{j-1}$ whose image in $H_I^{j-1}(R)/(f_1, f_2)H_I^{j-1}(R)$ is $[\eta]$. Then $[\eta_e] \in H_I^{j-1}(R)$ belongs to $\mathbf{F}^e(H)$. Hence $[\eta_e] \in \mathbf{F}^e(L)$; that is, there are $\tau_e \in \check{C}_e^j$, $\alpha_{1,e}, \alpha_{2,e} \in \check{C}_e^{j-1}$ and $\beta_{1,e}, \beta_{2,e} \in \ker(\delta_e^j)$ such that

$$\delta_e^j(\tau_e) = 0,\ \delta_e^{j-1}(\alpha_{1,e}) = f_1^{p^e}\tau_e,\ \delta_e^{j-1}(\alpha_{2,e}) = f_2^{p^e}\tau_e$$

and that

$$\begin{aligned}
[\eta_e] &= \varphi_{2,e}^{0,j}([\tau_e]) \\
&= \overline{[-f_2^{p^e}\alpha_{1,e} + f_1^{p^e}\alpha_{2,e}]} + f_1^{p^e}\beta_{1,e} + f_2^{p^e}\beta_{2,e} \\
&= \overline{[-f_2(f_2^{p^e-1}\alpha_{1,e}) + f_1(f_1^{p^e-1}\alpha_{2,e})]} + f_1(f_1^{p^e-1}\beta_{1,e}) + f_2(f_2^{p^e-1}\beta_{2,e})
\end{aligned}$$

Set $\widetilde{\tau} = f_1^{p^e-1}f_2^{p^e-1}\tau_e$, $\widetilde{\alpha}_1 = f_2^{p^e-1}\alpha_{1,e}$ and $\widetilde{\alpha}_2 = f_1^{p^e-1}\alpha_{2,e}$. Then

$$\delta_e^j(\widetilde{\tau}) = 0,\ \delta_e^{j-1}(\widetilde{\alpha}_1) = f_1\widetilde{\tau},\ \delta_e^{j-1}(\widetilde{\alpha}_2) = f_2\widetilde{\tau}$$

and

$$[\eta_e] = \overline{[-f_2(f_2^{p^e-1}\alpha_{1,e}) + f_1(f_1^{p^e-1}\alpha_{2,e})]} + f_1(f_1^{p^e-1}\beta_{1,e}) + f_2(f_2^{p^e-1}\beta_{2,e})$$

$$= \overline{[-f_2\widetilde{\alpha}_1 + f_1\widetilde{\alpha}_2]} + f_1(f_1^{p^e-1}\beta_{1,e}) + f_2(f_2^{p^e-1}\beta_{2,e})$$

$$= d_2^{0,j}(\widetilde{\tau})$$

This proves that $[\eta_e]$ is in the image of $d_2^{0,j}$. This completes the proof. □

Combining Theorem II.4, Theorem III.6, Theorem III.9, and Theorem III.10, the following theorem is immediate:

Theorem III.11. Let R be a Noetherian regular ring of prime characteristic p. If $f_1, f_2 \in R$ form a regular sequence in R, then

$$\operatorname{Supp}(H_I^j(\frac{R}{(f_1, f_2)}))$$

is Zariski-closed for each ideal I and each integer j.

The following corollary is immediate.

Corollary III.12. Let R be a Noetherian commutative ring of prime characteristic p that has finitely many isolated singular points. Let $f_1, f_2 \in R$ be a regular sequence. Then $H_I^j(R/(f_1, f_2))$ is Zariski-closed for each integer j and each ideal I.

IV

DIFFERENTIAL GRADED ALGEBRAS AND UNIVERSAL RESOLUTIONS

This chapter aims to provide a synopsis on DG algebras, as well as some basic constructions of universal resolutions. It concludes with a description of the cohomological support variety which will be instrumental in Chapter VI and Chapter VII. Section 1 provides an overview of DG algebras, and Section 2 considers a few relevant universal resolutions and defines the cohomological support variety, providing a re-proof of the equivalence of two relevant constructions.

1. Introduction to differential graded algebras

A more in-depth rundown on the basics of DG algebras can be found in [Avr98]. This section pulls largely from this and [Pol19].

Definition IV.1. A *DG algebra* A over a ring Q consists of a chain complex, which we also refer to as A, and a *multiplication stucture*, an unitary and associative operator $a \otimes b \mapsto ab$ on $A \otimes A$ comprising a function

$$\cdot : A \otimes A \to A$$

such that

$$ab = (-1)^{|a||b|} ba \text{ for } a, b \in A \quad \text{and} \quad a^2 = 0 \text{ when } |a| \text{ is odd,}$$

comprising *commutativity*, and the graded algebra A is concentrated in non-negative degree.

Remark IV.2. Note that some texts refer to this a *commutative DG algebra concentrated in non-negative degree*, and drop the commutativity and degree conditions for the term "DG algebras."

The DG algebra is a valuable tool for understanding and manipulating sufficiently nice resolutions, especially between rings, such as a resolution over some base ring Q of a module over a quotient ring R. Unfortunately a DG algebra resolution does not always exist:

Theorem IV.3 (see [Avr98, Theorem 2.3.1])**.** Let k be a field, and Q be the polynomial ring $k[s_1, s_2, s_3, s_4]$ with the usual grading, or the power series ring $k[\![s_1, s_2, s_3, s_4]\!]$. There exists no DG algebra structure on the minimal Q-free resolution U of the residue ring $R = Q/I$ where

$$I = (s_1^2, s_1 s_2, s_2 s_3, s_3 s_4, s_4^2)$$

or on the minimal Q-free resolution U' of the Cohen-Macaulay residue ring $R' = Q/I'$ where

$$I' = I + (s_1 s_3^6, s_2^7, s_2^6 s_4, s_3^7).$$

However, some large classes of resolutions are known to have DG algebra structures:

Theorem IV.4 (see [Avr98, Proposition 2.1.4])**.** If A is a projective resolution of a Q-module R, such that $A_0 = Q$ and $A_n = 0$ for $n \geq 4$, then A has a structure of DG algebra.

Though in this paper we primarily will concern ourselves with DG algebras, many of the techniques we will use will be extensible to even more resolutions than those we will work with, and we will attempt to signpost these throughout. In order to extend this kind of work, we can consider complexes also which, though they may not have DG algebra structures, still have avatars of DG algebras within them:

Definition IV.5. A *DG module* M over a DG algebra A consists of a chain complex with underlying graded module M and differential ∂, along with an *A-multiplication structure*, an operator $a \otimes m \to am$ on $A \otimes M$ comprising a function

$$A \otimes M \to M$$

such that

$$\partial(am) = \partial(a)m + (-1)^{|a|}a\partial(m)$$

and which is unitary and associative for their natural definitions.

Let A be a DG Q-algebra and B be a DG A-module. $A^\natural$ is the underlying graded Q-algebra of A and $B^\natural$ is the underlying graded $A^\natural$-module of B. Furthermore, the *suspension* $\Sigma^i B$ of a DG A-module B for some integer i is given by

$$(\Sigma^i B)_n = B_{n-i}, \quad \partial^{\Sigma^i B} = (-1)^i \partial^B \quad \text{and} \quad a \cdot \Sigma^i b = (-1)^{|a|i}\Sigma^i(ab).$$

We define the suspension of a chain complex which is not a DG module by imposing only the first two of these three conditions.

A DG A-module P is *semiprojective* if for every morphism of DG A-modules $\alpha : P \to N$ and each surjective quasi-isomorphism of DG A-modules $\gamma : M \to N$ there exists a unique-up-to-homotopy morphism $\beta : P \to M$ of DG A-modules such that $\alpha = \gamma\beta$. A DG A-module F is *semifree* if $F^\natural$ is a free $A^\natural$-module. A *semiprojective (resp. semifree) resolution* is a quasi-isomorphism $P \to M$ of DG A-modules such that P is semiprojective (resp. semifree).

Theorem IV.6 (see [Avr98, Propositions 1.3.2 and 1.3.3])**.** Say F, G, M, N are DG A-modules with F, G semifree. If we have a quasi-isomorphism $\gamma : F \to G$, then $\gamma \otimes M$ is a quasi-isomorphism as well. If we have a quasi-isomorphism $\beta : M \to N$, then $F \otimes \beta$ is a quasi-isomorphism as well.

Remark IV.7. Recall that there is a sign inherent to the tensor of two maps: if f, g are endomorphisms of M, N respectively, then

$$f \otimes g := M \otimes N \to M \otimes N, \qquad (m \otimes n) \mapsto (-1)^{|g|(\textit{entry of } m)}(f(m) \otimes g(n)).$$

Definition IV.8. A *quasi-isomorphism* between DG algebras A and A' is a collection of DG algebras $A =: A_0, A_1, \cdots, A_k := A'$ along with for each $1 \leq i \leq k$ a map f_i either from A_{i-1} to A_i or vice-versa which induces an isomorphism on homology.

2. Differential graded algebras and base change

This section is based on material from [Avr98, Chapter 3] and [Pol21].

2.1. Universal Resolutions. We begin a synopsis of various results regarding universal resolutions. [BCL+25] is not only an excellent collection of results of this kind but a treatise on the mechanisms underlying these universal resolutions in various contexts, and anyone interested in understanding the properties invariant or detected by universal resolutions, especially in a more general context, should read it. In the context of DG algebras and DG algebra resolutions specifically, we most often seek uniformity such that, by applying simple DG algebra-flavored modifications to our resolutions, most notably tensoring by some DG algebra assuming they have a DG module structure, we can achieve this base change. In this section, say we have an R-module M with a not-necessarily-semifree A-resolution F, where A is a DG algebra resolution of R over Q.

2.1.1. *The bar construction.* This section covers one of the most fundamental universal resolutions, the bar construction.

Construction IV.9 (bar construction)**.** Let $\tilde{A}$ be the cokernel of the map $Q \to A, 1 \mapsto 1_A$. Consider the double complex

$$\begin{array}{c} \vdots \\ \downarrow \\ A \otimes_Q \tilde{A} \otimes_Q \tilde{A} \otimes_Q F \\ \downarrow \\ A \otimes_Q \tilde{A} \otimes_Q F \\ \downarrow \\ A \otimes_Q F \\ \downarrow \\ F \\ \downarrow \\ 0 \end{array}$$

where our vertical differential is the map

$$A \otimes_Q \underbrace{\tilde{A} \otimes_Q \cdots \otimes_Q \tilde{A}}_{n \text{ copies}} \otimes_Q F \to A \otimes_Q \underbrace{\tilde{A} \otimes_Q \cdots \otimes_Q \tilde{A}}_{n-1 \text{ copies}} \otimes_Q F,$$

$$a \otimes \tilde{a}_1 \otimes \cdots \otimes \tilde{a}_n \otimes f \mapsto (a\tilde{a}_1) \otimes \tilde{a}_2 \otimes \cdots \otimes \tilde{a}_n \otimes f$$
$$+ \sum_{i=1}^{n-1} (-1)^{i-1} a \otimes \tilde{a}_1 \otimes \cdots \otimes (\tilde{a}_i \tilde{a}_{i+1}) \otimes \cdots \otimes a_n \otimes f$$
$$+ (-1)^n a \otimes \tilde{a}_1 \otimes \cdots \otimes \tilde{a}_{n-1} \otimes (\tilde{a}_n f).$$

The totalization of this double complex is quasi-isomorphic to F and is known as the *bar construction* $B(F)$.

Since $B(F)$ is semi-free over A, by Theorem IV.6 $B(F) \simeq B(F) \otimes_A R$, yielding the following

Theorem IV.10 (see [Avr98, Theorem 3.1.1])**.** If F is an A-resolution of M, then $B(F)$ is an A-resolution of M such that $B(F) \otimes_A R$ is an R-resolution of M.

Thus, given any A-resolution of M, we can by way of a DG algebra tensor construct an R-resolution of M from it, giving us our first universal resolution.

2.1.2. *Children of a resolution described by Shamash.* These constructions will be closely related to the DG algebra E, which is a function of the expression $R = Q/I = Q/(\boldsymbol{f})$ of a ring R for Q some ring and I the ideal generated by some sequence $\boldsymbol{f} = (f_i)$, where $n := |\boldsymbol{f}|$: in particular, E is defined as the DG algebra

$$Q[\xi_1, \ldots, \xi_n | \partial \xi_i = f_i].$$

A *Koszul resolution* of a DG E-module M is a map of DG E-modules $P \to M$ which, as a map of DG Q-modules via restriction of scalars $Q \to E$, is a semiprojective resolution.

Say first that R is a ring quotient Q/I where I is generated by a regular sequence $\boldsymbol{f} = (f_i)$. In this case, Gulliksen [Gul74] noted that if M and N are R-modules, then $\mathrm{Ext}^*_R(M, N)$ naturally inherits the structure of a graded module over a polynomial ring with variables χ_i of cohomological degree 2 (that is, there are natural actions on the graded module $\mathrm{Ext}^*_R(M, N)$ corresponding to the f_i's which have degree -2). This structure comes from handling connecting homomorphisms related to a Koszul resolution of M. This graded module could be called the first instance of a *cohomological support module*, that is, any representation of

Ext as a module over a polynomial ring with variables given by automorphisms in this way, though this name and its usage in providing invariants to modules like M and N came later.

In [AB00a], a more computationally-explicit way of constructing these modules was discovered, which we describe here, assuming that we also had access to an explicit Koszul resolution F of M. This construction yields a DG E-module resolution of M, which, when tensored over E with R, yields a famous resolution described by Shamash [Sha69]. Let Γ be the polynomial ring $Q[y_1, \ldots, y_n]$, where each y_i has homological degree 2. As before, we wish to make a cohomological support module with an explicit action given by the graded ring $\mathscr{S} = \operatorname{Hom}(Q[\chi_1, \ldots, \chi_n], Q)$, where each χ_i has homological degree -2. To do so, we will define an $\mathscr{S}$-action on Γ, and our $\mathscr{S}$-action on our eventual resolution will be induced by that on Γ. In particular, we define

$$\chi_j y^i = \begin{cases} y_1^{i_1} \cdots y_j^{i_j - 1} \cdots y_n^{i_n} & \text{when } i_j > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $i = (i_1, \ldots, i_n)$ is any sequence of non-negative integers, $j \in \mathbb{Z}_{\geq 0}$, and $y^i = y_1^{i_1} \cdots y_n^{i_n}$. The construction begins with a complex referred to as L, defined as follows:

$$L^\natural = E \otimes \Gamma \otimes E$$

$$\partial_L = 1 \otimes y^i \otimes 1 \mapsto \sum_{1 \leq j \leq n} 1 \otimes \chi_j y^i \otimes \xi_j - \xi_j \otimes \chi_j y^i \otimes 1,$$

where $i = (i_1, \ldots, i_n)$ is any sequence of non-negative integers and $y^i = y_1^{i_1} \cdots y_n^{i_n}$. This is then endowed with the structure of an $E \otimes_Q E$ module: $a \otimes y \otimes b$ is considered to be $(a \otimes b) \otimes y$, and this is an E module via the copies of E on the left and the right via the construction of a tensor of DG algebras given in [Avr98]: in particular, multiplication for the left copy of E is given with no sign, that is, $(a \otimes 1 \otimes 1)(b \otimes y \otimes c) = (ab \otimes y \otimes c)$, whereas multiplication for the right copy of E is given with a sign the product of the entries of the multiplicand and the left-hand E entry, that is, $(1 \otimes 1 \otimes a)(b \otimes y \otimes c) = (-1)^{|a||c|}(b \otimes y \otimes ac)$.

This yields the differential

$$\begin{aligned}\partial_L(a\otimes y^i\otimes b) &= \partial a\otimes y^i\otimes b+(-1)^{|a|}a\otimes y^i\otimes\partial b\\ &\quad+(-1)^{|a|+|b|}\sum_j a\otimes\chi_j y^i\otimes b\xi_j-(-1)^{|a|}\sum_j a\xi_j\otimes\chi_j y^i\otimes b\\ &=\partial a\otimes y^i\otimes b+(-1)^{|a|}a\otimes y^i\otimes\partial b\\ &\quad+(-1)^{|a|}\sum_j a\otimes\chi_j y^i\otimes\xi_j b-\sum_j\xi_j a\otimes\chi_j y^i\otimes b \qquad \text{(IV.1)}\end{aligned}$$

where a and b are basis elements of the free graded R-module E and i is as before. This can also be written (as it is by Pollitz) as

$$\partial_L=\partial_E\otimes 1\otimes 1+1\otimes 1\otimes\partial_E+\sum_j\left(1\otimes\chi_j\otimes\xi_j-\xi_j\otimes\chi_j\otimes 1\right).$$

Remark IV.11. This module-over-a-tensor-product structure is written slightly differently in [Pol21] than it is in [AB00a], though their structures are the same: when considering L as a module over a tensor product, Pollitz considers the left E-action on the right copy of E in $L^\natural$ to be a right action on L, so he considers us to have made a $E\otimes E^{\mathrm{op}}$-module. However, he then later considers this left action on the right copy of E as our left E-action on L when constructing $L\otimes_E F$.

We then consider the tensor $L\otimes_E F$, our Koszul resolution of M, considering L to be an E-module via our *right* copy of E. This tensor, which we will call $U_E(F)$, is then considered as an E-module itself via the *left* copy of E.

Theorem IV.12 ([AB00a, Proof of Theorem 2.4])**.** The complex $U_E(F)$ is a DG E-module resolution of M.

By the definition of the tensor product of E-modules and (IV.1), such a DG module must have differential.

$$\partial_{U_E(F)}(e\otimes y^i\otimes f)=\partial_L(e\otimes y^i\otimes 1)\otimes f+(-1)^{|e|+|y^i|}(e\otimes y^i\otimes 1)\otimes\partial f$$

$$
\begin{aligned}
&= \partial e \otimes y^i \otimes f + (-1)^{|e|} \sum_j (e \otimes \chi_j y^i \otimes \xi_j) \otimes f \\
&\qquad - \sum_j \xi_j e \otimes \chi_j y^i \otimes f + (-1)^{|e|+|y^i|} (e \otimes y^i \otimes 1) \otimes \partial f \\
&= \partial e \otimes y^i \otimes f + \sum_j \xi_j (e \otimes \chi_j y^i \otimes 1) \otimes f \\
&\qquad - \sum_j \xi_j e \otimes \chi_j y^i \otimes f + (-1)^{|e|+|y^i|} (e \otimes y^i \otimes 1) \otimes \partial f \\
&= \partial e \otimes y^i \otimes f + (-1)^{|e|+|\chi_j y^i|} \sum_j e \otimes \chi_j y^i \otimes \xi_j f \\
&\qquad - \sum_j \xi_j e \otimes \chi_j y^i \otimes f + (-1)^{|e|+|y^i|} e \otimes y^i \otimes \partial f \\
&= \partial e \otimes y^i \otimes f + (-1)^{|e|} \sum_j e \otimes \chi_j y^i \otimes \xi_j f \\
&\qquad - \sum_j \xi_j e \otimes \chi_j y^i \otimes f + (-1)^{|e|} e \otimes y^i \otimes \partial f.
\end{aligned}
$$

This can be written also, as it is in [Pol21], as

$$
\partial_E \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \partial_F + \sum_j (1 \otimes \chi_j \otimes \xi_j - \xi_j \otimes \chi_j \otimes 1).
$$

Since R is a complete intersection, the augmentation $E \to R$ is a quasi-isomorphism, so by [Avr98, Proposition 1.3.2] $U_E(F) \otimes_E R$ is an R-resolution of M, which is the universal resolution of M defined in [AB00a]. $U_E(F)$ is referred to in [Pol21] as a universal resolution of M as well—here we will refer to it as our *universal E-resolution of M given by F.* We summarize with the following

Definition IV.13. When R is a quotient of some commutative ring R by some sequence $\boldsymbol{f}$, $\operatorname{Ext}_E(M, N)$ may be interpreted as an $\mathscr{S}$-module via the constructions above, where $\mathscr{S}$ has $|\boldsymbol{f}|$ generators.

Although $U_E(F)$ is originally described in [AB00a], it is only advertised as useful-in-itself in [Pol21], where it was explicitly determined that the resulting $\mathscr{S}$-module, given by its action on the underlying copy of Γ, was independent of choice of Koszul resolution. This construction has the advantange over the Gulliksen construction of having our χ_i act on a much smaller portion of our graded module, simplifying any computations related to them. This universal resolution can also be shown to be of the same ilk as the other as seen in [BCL$^+$25], allowing for it to be understood via the analogues and generalizations made therein.

It is implicitly noted in [Pol21] and [AG02] that, although we need $\boldsymbol{f}$ to be a complete intersection in order to construct an R-resolution via the above methods, this condition is not necessary to construct our universal E-resolution. In fact, it is shown in [Pol19] that whenever we have two DG modules M, N over some DG algebra A, Ext between them can be shown to have a $\mathrm{Ext}_A(N,N)$-$\mathrm{Ext}_A(M,M)$ bimodule structure between them, so our construction is primarily a consideration of a special case of a larger concept.

We complete this section by recalling the explicit descriptions of the cohomological support module for E-modules M, N, and proving one notable one. This requires us first to introduce a bit of notation regarding operations on Hom complexes and tensors of complexes, which can be found in [CFH24]:

Definition IV.14. Let M, N be complexes of modules. Then

$$\mathrm{Hom}(M,N) = \bigoplus_{j-i} \mathrm{Hom}(M_i, N_j), \partial_{\mathrm{Hom}(M,N)}(f) = \partial_N \circ f - (-1)^{j-i} f \circ \partial_M$$

Definition IV.15. If f, g are endomorphisms of M, N respectively, then

$$f \otimes g := M \otimes N \to M \otimes N, (m \otimes n) \mapsto (-1)^{|g|(\mathit{entry\ of\ m})}(f(m) \otimes g(n)).$$

Definition IV.16. If f, g are endomorphisms of M, N respectively, then

$$\mathrm{Hom}(f,g) := \vartheta \mapsto (-1)^{|f|(|g|+|\vartheta|)} g\vartheta f.$$

With these in mind, we can state the following

Theorem IV.17 ([Pol21, Proposition 4.2.8]).

$$\mathrm{Hom}_E(U_E(F), N) \cong \mathscr{C}_E(F, N),$$

and consequently

$$\mathrm{Ext}_E(M, N) \cong \mathrm{H}\left(\mathscr{C}_E(F, N)\right),$$

where

$$\begin{aligned}\mathscr{C}_E(F, N)^{\natural} &:= \mathscr{S} \otimes_Q \mathrm{Hom}_Q(F, N)^{\natural},\\ \partial^{\mathscr{C}_E(F,N)} &= 1 \otimes \partial^{\mathrm{Hom}_Q(F,N)} + \sum_{i=1}^{n}(\chi_i \otimes \mathrm{Hom}(\xi_i, N) - \mathrm{Hom}(F, \xi_i)),\end{aligned}$$

ξ_i connoting left-multiplication on the E-module on which it acts.

Remark IV.18. Say M, N are R-modules, which may be considered as DG E-modules by considering their multiplication by our e_i terms to be zero. Then the expression of $\mathscr{C}_E(F, N)$ described above can be simplified to an expression of $\mathrm{Ext}_E(M, N)$ which can be found in [AG02]. As already stated, if R is a complete intersection, then this is isomorphic to $\mathrm{Ext}_R(M, N)$. This is how Macaulay2 [GS] calculates `Ext(M,N)` for R a complete intersection. In fact, when you enter the command `Ext(M,N)` into Macaulay2, the resulting class is `Module` and the ring over which it lies is $\mathscr{S}$.

We will provide a re-proof of Theorem IV.17 at the end of this section. For the remainder of this chapter, we will consider Q/I to be a *minimal regular presentation* of R, so that either Q is regular local with residue field k and I is in the square of its maximal ideal or Q is a polynomial ring over a field k and I is generated by forms of degree at least 2. More specifically we will assume $N = k$. In this case, the above reduces to the following

Corollary IV.19.

$$\mathrm{Ext}_E(M, k) \cong \mathrm{H}\left(\mathscr{C}_E(F)\right)$$

where

$$\mathscr{C}_E(F)^\natural := \mathscr{S} \otimes_k \operatorname{Hom}_Q(F,k)^\natural \quad \text{with} \quad \partial^{\mathscr{C}_E(F)} = 1 \otimes \partial^{\operatorname{Hom}_Q(F,k)} + \sum_{i=1}^{n} \chi_i \otimes e_i \,. \tag{IV.2}$$

2.2. Cohomological support varieties. A cohomological support variety is a homological invariant defined in [Avr89] based on the cohomological support module above. It can be found in many forms [Avr98, AB00a, AB00b, AG02] and has been used to prove various homological properties [Avr98, AB00b, BGP21, BGP24, Pol19, Ste13]. In particular, it is a modification of the support of the module:

Definition IV.20. Let A be a graded ring and M be a graded A-module. Then the *cohomological homogeneous support* $\operatorname{Supp}^+(M)$ of M consists of the prime ideals of $\operatorname{Supp}(M)$ not containing A^+, the ideal of all cohomologically-positively-graded elements of A.

Definition IV.21. The *cohomological support variety* $V_R(M)$ of a module M is the cohomological homogeneous support of the $\mathscr{S} \otimes_Q k$-module $\operatorname{Ext}_E(M,k)$:

$$V_R(M) := \mathit{Supp}^+_{\mathscr{S}\otimes_Q k}\left(\operatorname{Ext}_E(M,k)\right).$$

We will also use $\mathrm{V}_R(M)$ to refer to the support of $\operatorname{Ext}_E(M,k) \otimes_k \overline{k}$, where $\overline{k}$ is an algebraic closure of k which we fix henceforth—these are determinable from each other by the Nullstellensatz. It is also worth noting that by [BGP25, 1.2.2], if $\mathrm{H}(M)$ is finitely generated over R, there exists some such F which is a bounded complex of free Q-modules, implying that $\mathscr{C}_E(F)$ is a finite rank free graded $\mathscr{S}$-module, and furthermore that $\operatorname{Ext}_E(M,k)$ is finitely generated over $\mathscr{S}$.

2.3. A re-proof of Theorem IV.17. Here we re-prove Theorem IV.17, which will hopefully be illuminating for those seeking to work intimately with these sorts of constructions. For a a map of graded modules or a homogeneous element of a graded module, let $|a|$ be its degree. Before proving this theorem, we must prove one lemma regarding our flexibility in sign convention:

Lemma IV.22. Consider a graded module G over $\mathscr{S}$ and endow G with a new $\mathscr{S}$-module structure to produce a new module G' in which the Q-module structure is preserved and multiplication is given by

$$\chi^i g' = c^{|\chi^i|}\chi_i g$$

where g is the element of G corresponding to $g' \in G'$ where i is some multi-index and c is a unit in Q. Then $G \cong G'$.

Proof. The map $g \mapsto c^{-|g|}g'$, where g' and g are related as above, is an isomorphism. □

We now have the necessary tools to prove Theorem IV.17:

Proof of Theorem IV.17. $\mathrm{Hom}^\bullet_E(U_E(F), N)$ is the subcomplex of $\mathrm{Hom}^\bullet_Q(U_E(F), N)$ given by the condition of E-linearity. By Definition IV.14 it has differential

$$\pi \mapsto \partial_N \pi - (-1)^{|\pi|}\pi\partial_{U_E(F)},$$

which by our prior discussion of $U_E(F)$ is

$$\pi \mapsto \partial_N\pi - (-1)^{|\pi|}\pi\left(\partial_E + (-1)^{|e|}\partial_F + \sum_j \left((-1)^{|e|}(\chi_j)_\Gamma(\xi_j)_F - (\xi_j)_E(\chi_j)_\Gamma\right)\right),$$

where $|\pi|$ is the degree of π, and under our natural expression of $(U_E(F))^\natural$ as $(E\otimes_Q \Gamma \otimes_Q F)^\natural$, $|e|$ is the degree of our entry in E and for $C \in \{E, \Gamma, F\}$ and g some action on C, we use g_C to refer to the action on U which applies g to C. Now by E-linearity of π, $\pi(\xi_j)_E = (-1)^{|\pi|}\xi_j\pi$, allowing us to write our differential as

$$\pi \mapsto \partial_N\pi + \sum_j \left(\xi_j\pi(\chi_j)_\Gamma - (-1)^{|\pi|+|e|}\pi(\chi_j)_\Gamma(\xi_j)_F\right) - (-1)^{|\pi|}\pi\left(\partial_E + (-1)^{|e|}\partial_F\right).$$

Since $U_E(F)$ is a semi-free E-module we have

$$\begin{aligned}(\mathrm{Hom}^\bullet_E(U_E(F), N))^\natural &\cong (\mathrm{Hom}^\bullet_Q(U_E(F)\otimes_E Q, N))^\natural \\ &\cong ((U_E(F)\otimes_E Q)^* \otimes_Q N)^\natural, \\ &\cong (\mathscr{S}\otimes_Q F^* \otimes_Q N)^\natural,\end{aligned}$$

where the latter inherits an E-module structure from N, and $\gamma^* \otimes f^* \otimes n$ represents the E-linear map which takes $1 \otimes \gamma \otimes f$ to n, under our natural expression of $(U_E(F))^\natural$ as $(E \otimes_Q \Gamma \otimes_Q F)^\natural$. Note also, of course, that $\mathscr{S} = \Gamma^*$. Under this specification, if $\pi = \gamma^* \otimes f^* \otimes n$, note that pre-composing π with χ_j will replace γ^* to $(y^j\gamma)^*$, which, under our notation of $\mathscr{S}$, would be said to be replacing γ^* to $\chi_j\gamma^*$. Furthermore, pre-composing π with g_F for some will action of g on F will replace f^* with $(gf)^*$. Finally, $\partial_N\pi \pm \pi\partial_E$, the collection of all of the maps acting on the copy of E generated by $1 \otimes \gamma \otimes f$, will replace n with ∂n, as by E-linearity we can track only the image of $1 \otimes \gamma \otimes f$, which ∂_E takes to zero. Thus we have

$$
\begin{aligned}
&(\operatorname{Hom}^\bullet_E(U_E(F), N))^\natural \cong (\mathscr{S} \otimes_Q F^* \otimes_Q N)^\natural, \\
&\qquad \partial = \gamma^* \otimes f^* \otimes n \mapsto \gamma^* \otimes f^* \otimes \partial n - (-1)^{|n|-|\gamma|-|f|}\gamma^* \otimes (\partial f)^* \otimes n \\
&\qquad\qquad + \sum_j \left(\chi_j\gamma^* \otimes f^* \otimes \xi_j n - (-1)^{|n|-|\gamma|-|f|}\chi_j\gamma^* \otimes (\xi_j f)^* \otimes n\right) \\
&(\operatorname{Hom}^\bullet_E(U_E(F), N))^\natural \cong (\mathscr{S} \otimes_Q \operatorname{Hom}_Q(F, N))^\natural, \\
&\qquad \partial = \gamma^* \otimes \pi \mapsto \gamma^* \otimes \partial_N\pi - (-1)^{|\pi|-|\gamma|}\gamma^* \otimes (\pi\partial_F) \\
&\qquad\qquad + \sum_j \left(\chi_j\gamma^* \otimes \xi_j\pi - (-1)^{|\pi|-|\gamma|}\chi_j\gamma^* \otimes \pi\xi_j\right)
\end{aligned}
$$

As $|\gamma|$ is always even we have

$$
\begin{aligned}
&(\operatorname{Hom}^\bullet_E(U_E(F), N))^\natural \cong (\mathscr{S} \otimes_Q \operatorname{Hom}_Q(F, N))^\natural, \\
&\qquad \partial = \gamma^* \otimes \pi \mapsto \gamma^* \otimes \partial_{\operatorname{Hom}_Q(F,N)}\pi \\
&\qquad\qquad + \sum_j \left(\chi_j\gamma^* \otimes \xi_j\pi - (-1)^{|\pi|}\chi_j\gamma^* \otimes \pi\xi_j\right).
\end{aligned}
$$

By Definition IV.16 we have

$$
\begin{aligned}
&(\operatorname{Hom}^*_E(U_E(F), N))^\natural \cong (\mathscr{S} \otimes_Q \operatorname{Hom}_Q(F, N))^\natural, \\
&\qquad \partial = \gamma^* \otimes \pi \mapsto \gamma^* \otimes \partial_{\operatorname{Hom}_Q(F,N)}\pi \\
&\qquad\qquad + \sum_j \left(\chi_j\gamma^* \otimes (\operatorname{Hom}(F, \xi_j) - \operatorname{Hom}(\xi_j, N))(\pi)\right).
\end{aligned}
$$

Finally, by Definition IV.15 and the fact that $|\gamma|$ is always even we have

$$(\operatorname{Hom}^*_E(U_E(F), N))^\natural \cong (\mathscr{S} \otimes_Q \operatorname{Hom}_Q(F, N))^\natural,$$
$$\partial = 1 \otimes \partial_{\operatorname{Hom}_Q(F,N)} + \sum_j (\chi_j \otimes (\operatorname{Hom}(F, \xi_j) - \operatorname{Hom}(\xi_j, N))).$$

These two summands are added, rather than subtracted as in Theorem IV.17. To complete our proof, we appeal to Lemma IV.22. □

V

FORMALITY, GOLODITY, AND A_∞ ALGEBRAS VIA MASSEY PRODUCTS

In Section 1, we provide various characterizations of and relevant background on Golodity. In Section 2, we provide relevant background on Massey products, and A_∞ algebras. In Section 3, we use the lens of Massey products to both understand formality and Golodity generally, to more explicitly prove Fact I.3, and to better describe the relationship between Fact I.2 and Fact I.3. In Section 4, we express hope regarding the future of this line of inquiry, particularly in a computational context.

We will consider $(R, \mathfrak{m}, k)$ a local ring, and let K^R be the DG algebra given by the Koszul complex of some minimal generating set of $\mathfrak{m}$. Before we begin, we provide a few definitions, including of some of the terms given in the introduction, so as to illuminate their relationship to this chapter. These are taken in part from [BCL$^+$25] but are relatively well-known.

Definition V.1. An *augmented* (DG) k-algebra A is one which has a given homomorphism to some ring, which for our purposes will always be k

Definition V.2 (Koszul algebra, [BCL$^+$25, Definition 2.1])**.** An augmented k-algebra K is Koszul (over k) if it admits a grading $K = \oplus_{w>0} K_{(w)}$, known as a *weight grading*, such that $K_{(0)} = k$ and such that the minimal resolution of k over K is linear with respect to this grading.

Definition V.3 (Formality, [BCL$^+$25, 2.3])**.** A DG k-algebra is *formal* if it is quasi-isomorphic to $\mathrm{H}(K)$.

Definition V.4 (Koszul DG algebra, [BCL$^+$25, Definition 2.4])**.** A DG algebra K is *Koszul* if it is formal and $H(K)$ is Koszul as a graded algebra.

Definition V.5 (Koszul homomorphism, [BCL+25, Definition 2.6])**.** A finite local homomorphism $Q \to R$ is *Koszul* if, for A a DG Q-algebra resolution of R, $A \otimes_Q k$ is a Koszul DG k-algebra.

Recall as we mentioned in the introduction that the last of these is highly relevant to a number of universal resolutions, and note here its close relationship with formality.

1. Characterizations of Golodity

Golodity is an extremal local ring property of the resolution of the residue field. Let $Q \to R$ be a finite local homomorphism, and for M a finite R-module let $P_M^R(t)$ be its Poincaré series. It is shown in [Avr86] that

$$P_M^R(t) \preccurlyeq \frac{P_M^Q(t)}{1 - t(P_R^Q(t) - 1)}, \tag{V.1}$$

where $\preccurlyeq$ indicates simultaneous inequalities in each pair of coefficients of the same degree. It is noted further in, for example, [Avr98] that the right hand side is independent of our choice of Q, and in fact, by using completions and the Cohen structure theorem, it is shown that it does not require the existence of Q at all. Here, we provide this version for the specific case $M = k$, which can be found for example at [Avr98, (5.0.1)]:

$$P_k^R(t) \preccurlyeq \frac{(1+t)^{\operatorname{embdim}(R)}}{1 - \sum_{j=1}^{\operatorname{codepth}(R)} \operatorname{rank}_k \operatorname{H}_j(K^R) t^{j+1}}, \tag{V.2}$$

Definition V.6. A local ring R is *Golod* if equality holds in (V.2).

One additional definition of Golodity is given using *trivial Massey operations*, or the closely related *Massey products*. (What we refer to as Massey products here are referred to as *ordinary Massey products* elsewhere to distinguish them from the more general *matric Massey products*, in which our elements are matrices of homology classes, rather than individual homology classes themselves.)

Definition V.7 (Massey product)**.** Let A be a DG algebra. Say that $h_1, \ldots, h_n \in H(A)$. Consider the possible sets of $a_{i,j}$ for $0 \leq i \leq n-1, i+1 \leq j \leq n$ such that

- $\forall 1 \leq i \leq n, \partial\mu(a_{i-1,i}) = 0$ and $\mu(a_{i-1,i}) - a_i = 0 \in \mathrm{H}(A)$,
- $\forall 1 < j < n, \forall 0 \leq i \leq n-j, \partial a_{i,i+j} = \sum_{m=1}^{j-1} \overline{\mu(h_{i+1}, \ldots, h_{i+m})}\mu(h_{i+m+1}, \ldots, h_{i+j})$.

The *Massey product* $\langle a_1, \ldots, a_n \rangle$ is the space spanned by the possible homology classes

$$\sum_{m=1}^{j-1} \overline{\mu(h_1, \ldots, h_m)}\mu(h_{m+1}, \ldots, h_n),$$

Definition V.8 (Trivial Massey Operation, [Avr86, 1.4.2])**.** Let A be a DG algebra with $H_0(A) \cong k$. A *trivial Massey operation* on A is a k-basis $\boldsymbol{b}$ of $\mathrm{H}_{\geq 1}(A)$ along with a map $\mu : \bigsqcup_{p=1}^{\infty} \boldsymbol{b}^p \to A$ such that

- $\forall h \in \boldsymbol{b}, \partial\mu(h) = 0$ and $\mu(h) - h = 0 \in \mathrm{H}(A)$ (μ selects representatives of cycle classes),
- $\forall (h_1, \ldots, h_p) \in \boldsymbol{b}^p, \partial\mu(h_1, \ldots, h_p) = \sum_{j=1}^{p-1} \overline{\mu(h_1, \ldots, h_j)}\mu(h_{j+1}, \ldots, h_p)$,

where here $\overline{a} = (-1)^{|a|+1}a$.

Note that if $H_0(A) \cong k$, then we admit a trivial Massey operation if and only if all of our Massey products of elements of $H_{\geq 1}(A)$ are zero: Massey products are additive on their terms, so if we have a trivial Massey operation, then we can conclude from it trivial Massey products on other tuples of homology classes as well.

This paper provides another definition as well: a *Golod DG algebra* is defined as the existence of a trivial Massey operation, which, though it was given a slightly different definition, was the same on K^R. (This is referred to in [Avr86] simply as a *Golod algebra*.) He gave a collection of equivalent definitions of a Golod DG algebra, which, in the case of K^R, are equivalent to the Golodity of R, giving us the following:

Theorem V.9 (cf. [Avr86] and [Avr13, Theorem 11.2])**.** The following are equivalent:

- R is Golod,
- For some basis $\boldsymbol{b}$ of $\mathrm{H}_{\geq 1}(K^R)$, K^R admits a trivial Massey operation using $\boldsymbol{b}$,

- for any basis $\boldsymbol{b}$ of $\mathrm{H}_{\geq 1}(K^R)$, K^R admits a trivial Massey operation using $\boldsymbol{b}$,
- $\mathrm{H}_{\geq 1}(K^R)$ has all trivial Massey products,
- K^R is formal and $H(K^R)$ is a *trivial extension of* k: it has vector space $k \oplus \langle \boldsymbol{b} \rangle$ for some basis $\boldsymbol{b}$ of $\mathrm{H}_{\geq 1}(K^R)$, unit $(1,0)$, and multiplication structure $(a,v)(a',v') = (aa', av' + a'v)$.

Levin [Lev75] considers the *Golod homomorphism*, which is also defined so as to beget Golod rings, though in order to specify how we must pass through completion. Though we will not use the definition itself, we provide it here for completeness:

Definition V.10 (Golod homomorphism)**.** A local surjective homomorphism $Q \to R$ is *Golod* if the maps

$$\mathrm{Tor}_i^Q(k,k) \to \mathrm{Tor}_{i-1}^Q(k,\mathfrak{m}) \quad \text{and} \quad \mathrm{Tor}_i^Q(k,\mathfrak{m}) \to \mathrm{Tor}_i^R(k,\mathfrak{m})$$

induced by the short exact sequence $\mathfrak{m} \to R \to k$ are injective for all $i \geq 1$.

To relate Golod homomorphisms and Golod rings we need to consider $\widehat{R}$, the completion of R, as a proxy. Let $(R, \mathfrak{m}, k)$ be a local ring. A *regular presentation* of a ring is a surjective map $Q \to R$ where Q is a regular local ring. We say that it is *minimal* when $\mathrm{embdim}(R) = \mathrm{embdim}(Q)$, where embdim represents the embedding dimension. The Cohen structure theorem states that any complete ring has a minimal regular presentation, so say henceforth that $\widehat{R} = Q/I$, where $\widehat{\bullet}$ represents the completion and $(Q, \mathfrak{n}, k)$ is a regular local ring. Since $\widehat{R}$ is a faithfully-flat R-module, we can resolve modules $\widehat{M}$ over $\widehat{R}$ by tensoring resolutions of M over R with $\widehat{R}$, implying that the Poincaré series of M and $\widehat{M}$ are identical.

Theorem V.11 ([Lev75, Theorem 1.6])**.** If $Q \to R$ is a Golod homomorphism then equality is achieved in (V.1).

Theorem V.12 ([Lev75, Theorem 3.5])**.** If $Q \to R$ is a minimal regular presentation with Q a regular local ring such that the Massey product of any tuple of elements of $\mathrm{H}_{\geq 1}(K^R)$ exists and is zero, then $Q \to R$ is a Golod homomorphism.

Corollary V.13. If $Q \to R$ is a minimal regular presentation with Q a regular local ring such that K^R admits a trivial Massey operation, then $Q \to R$ is a Golod homomorphism.

Theorem V.14. R is Golod if and only if $Q \to \widehat{R}$ is a Golod homomorphism.

Proof. By Theorem V.11 and Corollary V.13, $Q \to \widehat{R}$ is Golod if and only if $\widehat{R}$ is Golod. By the fact that k must have the same Poincaré series whether expressed as an R-module or an $\widehat{R}$-module, (V.1) and the fact that embedding dimension, codepth, and the dimension of the homology of the Koszul complex are preserved under completion tells us that R is Golod if and only if $\widehat{R}$ is Golod. □

This is the definition of a Golod ring given in [Sch18].

2. Massey products and A_∞ algebras

We now have the tools to begin discussing the structure underlying both Golodity and formality: A_∞ algebras, which are closely related to Massey products, which we will explore in more depth in forthcoming sections. We begin with a basic explanation, which draws primarily from [Kel01].

Definition V.15. An A_∞*-algebra over* k is a graded k-vector space A with maps $m_i : A^{\otimes n} \to A$ of degree $i-2$ for positive integers i such that:

- $m_1^2 = 0$, that is, m_1 gives A a complex structure,
- $m_1 m_2 = m_2(1 \otimes m_1 + m_1 \otimes 1)$, so that m_2 is a multiplication operator that respects the Leibniz rule,
- $m_2(1 \otimes m_2 - m_2 \otimes 1) = m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$, that is to say, m_3 is a nullhomotopy of the deviance of our multiplication from associativity,

- for $n \geq 4$, we have

$$\sum_{\substack{r+s+t=n \\ s\geq 1}} (-1)^{r+st} m_{n+1-s} \left(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}\right) = 0.$$

Given the nature of m_1 and m_2, we may use ∂ to refer to m_1 on A (and to refer to

$$m_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes m_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes m_1$$

on $A^{\otimes n}$) and $a \cdot b$ to refer to $m_2(a, b)$ on occasion.

Note that a DG algebra can be naturally understood as an A_∞-algebra with $m_i = 0$ for $i > 2$, where m_1 is given by its differential and m_2 is given by its multiplication operation.

Definition V.16. A *morphism* of A_∞ algebras is a collection of maps (f_i) of degree $i - 1$ for positive integers i such that

- $f_1 \partial = \partial f_1$, that is, f_1 is a morphism of complexes,
- $\partial f_2 + f_2 \partial = f_1 m_2 - m_2(1 \otimes f_1 + f_1 \otimes 1)$, that is, f_2 is a nullhomotopy of the deviance of the respect of our multiplication structure,
- for $n \geq 3$,

$$\sum_{\substack{n=r+s+t \\ s\geq 1}} f_{n+1-s} \left(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}\right) = \sum_{\substack{r\leq n \\ i_1+\ldots+i_r=n}} (-1)^{v(i)} m_r \left(f_{i_1} \otimes \cdots \otimes f_{i_r}\right)$$

$$\text{where } v(i) = \sum_{j=1}^{i} (r-j)(i_j - 1).$$

Definition V.17 ([Pro11, Section 4.3])**.** A *quasi-isomorphism of A_∞ algebras* is a morphism (f_i) of A_∞ algebras such that f_1 is a quasi-isomorphism.

Theorem V.18 ([Kad82])**.** Two dg k-algebras are quasi-isomorphic if and only if they are quasi-isomorphic as A_∞ algebras.

Remark V.19. As [Kad82] appears to be lost, Theorem V.18 is taken from [BCL$^+$25, Proof of Proposition 5.4]. It seems that Kadeishvili was looking at a class of DG algebras known as connected commutative differential graded (CDG) algebras, but it's unclear—we were unable to show that connected CDG algebras quasi-isomorphic as DG algebras must also be quasi-isomorphic in the broader class of CDG algebras, though this would be sufficient, as removing excess copies of k at zero is simple.

Note that this does not imply that a quasi-isomorphism $A \to B$ of A_∞ algebras where A and B have no mu_n for $n \geq 3$ induces a quasi-isomorphism $A \to B$ of DG k-algebras!

Going forward, we will seek to impose A_∞ algebra structures on various objects. These objects will be abelian groups, often with either just a differential structure (such as with chain complex) which we wish to observe as m_1, or a differential strcuture and a multiplicative structure (such as with a DG algebra) which we will observe as m_2, such as a DG algebra, and we will seek to impose the remaining unspecified m_n term.

We end this section with two theorems regarding Golodity and formality with regards to A_∞ algebras which, while not directly used in the forthcoming discussion of the links between Golodity and formality, are notable for their similarity nonetheless. It would be interesting to see their similarity woven into the ensuing conversation.

Theorem V.20 ([BCL$^+$25, Propositions 5.1 and 5.4])**.** Let $Q \to R$ be a finite local homomorphism, let F be a minimal Q-free resolution of R, and let A be a DG Q-algebra resolution of R. One can assign to F an A_∞ algebra structure such that $F \to R$ is a quasi-isomorphism f of A_∞ algebras with $f_n = 0$ for $n > 1$. Under this assignment, $A \otimes_Q k$ is formal if and only if there is an A_∞ algebra structure μ which can be placed on F under the given conditions such that $\mu_n \otimes_Q k = 0$ for $n \geq 3$.

Theorem V.21 ([Bur15, Theorem 6.13])**.** $\widehat{R}$ (and thus R) is Golod if and only if there exists an A_∞ algebra structure μ which can be placed on a minimal free Q-resolution of R such that $\mu_n \otimes_Q k = 0$ for $n \geq 2$.

Note that $\mu_n \otimes_Q k$ will necessarily be zero for $n = 1$ as our resolution is minimal.

3. Formality and Golodity via A_∞ structures and Massey products

It would be enlightening if we could write some of our relationships here similarly to how those of formality are written: that is, whether there exist quasi-isomorphisms $H(A) \to A$ of A_∞ algebras for A some DG algebra, perhaps with some modifications. This would potentially allow us to apply some knowledge on A_∞ algebras to settings regarding their cousins the Massey products. When A is a DG algebra, we will assign to $H(A)$ the induced structure from A, in which μ_1 is the differential, μ_2 is induced by multiplication, and the remaining μ_n are zero, but on occasion we will speak on the existence of alternative higher μ_n structures on it, as is done elsewhere, so the reader is advised to keep their wits about them. We begin with the following

Theorem V.22 ([Kad05, Theorem 1])**.** Let A be a DG algebra with $H_i(A)$ free for all i. Then there is some A_∞ algebra structure ν which can be imposed upon $H(A)$ such that ν_i is induced from μ_i for $i \in \{1, 2\}$ and there is a quasi-isomorphism $H(A) \to A$ of A_∞-algebras.

This will be our primary weapon for understanding the relationship between Massey products and maps of A_∞ algebras surrounding $H(A)$ and A. Note that this condition of freedom primarily restricts the use of this theorem to DG k-algebras. However, this has notable members such as K^R with no assumptions imposed, and also includes any DG R-algebra whenever these rings contain k. We are going to be attempting to relate these structures to the existence or non-existence of certain Massey products, especially as they relate to Golodity and trivial Massey operations. With that in mind, we are often going to consider augmented DG k-algebras A and we will let $\overline{A}$ refer to $\ker(A \to k)$. This will allow us to (in fact, older references, such as [Avr86], define Massey products only on $\overline{A}$). Now consider the following

Theorem V.23 ([Val12, Proposition 7], Fundamental Theorem of $\boldsymbol{A_\infty}$ Quasi-Isomorphisms)**.** A DG k-algebra A is formal if and only if it has all trivial Massey products on three or more elements.

Note that from this, by Theorem V.18 and Theorem V.23, we have trivial Massey products on three or more elements if and only if there is a quasi-isomorphism $H(A) \to A$ of A_∞ algebras. We would like to use Theorem V.22 to prove results similar to this, in this case relating to Golodity. Noting that the Massey products on pairs of elements in an DG algebra A are exactly the products of elements in $H(A)$, there is the following obvious corollary of Theorem V.23:

Corollary V.24. R is Golod if and only if $\mathrm{H}\left(\overline{K^R}\right)$ has trivial multiplication and there is a quasi-isomorphism $\mathrm{H}\left(\overline{K^R}\right) \to K^R$ of A_∞ algebras.

We wish to re-prove this corollary using the construction of Theorem V.22.

Theorem V.25 ([BMFM19, Corollary 2.2])**.** For any DG k-algebra A and A_∞ structure ν on $H(A)$ such that ν_i is induced from μ_i for $i \in \{1, 2\}$, A has all trivial Massey products up to n elements if and only if $\nu_i = 0$ for $1 \leq i \leq n$.

Corollary V.26. A DG k-algebra A has all trivial Massey products on two or more elements if and only if $H(A)$ has a trivial multiplication structure and there is a quasi-isomorphism $H(A) \to A$ of A_∞ algebras.

Proof. The first condition yields that we always have trivial Massey products on two elements. By Theorem V.25, we have trivial Massey products on three or more elements if and only if some, or equivalently any, A_∞ algebra structure ν which can be imposed on $H(A)$ satisfying the conditions of Theorem V.22 is the natural one, and of course, if $H(A) \to A$ is a quasi-isomorphism under the natural A_∞ algebra structure of $H(A)$, then this natural structure satisfies the conditions of Theorem V.22. With this in mind, the second claim follows naturally from Theorem V.25. □

Corollary V.24 now naturally follows from Corollary V.26.

One may wish to generalize Theorem V.23 and Corollary V.24 and say that a DG k-algebra A has all trivial Massey products on n or more elements if and only if there is an A_∞ algebra structure ν which can be imposed on $H(A)$ such that there is a quasi-isomorphism $H(A) \to A$ of A_∞ algebras where $\nu_i = 0$ for $i \geq n$, again hoping to allow for the use of A_∞ algebras in contexts in which Massey products are most often used. However, this is not quite covered by Theorem V.25, which requires all lower Massey products to be zero in order to ensure recovery. It is claimed in [Kel01, 3.3] and [LPW09, Theorem 3.1] that there is a specific explicitly constructible choice of A_∞ structure on $H(A)$ such that Massey products are always recovered by our μ_n maps, but this was disproven in [BMFM19, 3]. A weaker version, however, is true:

Theorem V.27 ([BMFM19, Theorem A])**.** For any n-ary Massey product, there is some A_∞ structure ν on H satisfying the conditions of Theorem V.22 such that the image of ν_n contains it.

Consequently, A has all trivial Massey products on three or more elements if and only if for all possible choices of A_∞ structure on $H(A)$ following the conventions of Theorem V.22, ν_n is trivial for $n \geq 3$. That is to say, A has all trivial Massey products on three or more elements if and only if $H(A)$, given its natural A_∞ structure, is quasi-isomorphic to A but not quasi-isomorphic to any other A_∞ algebra structure on $H(A)$ which shares its ν_1 and ν_2 structures. A partial understanding of a way forward for vanishing of more than 3 elements can be found in [BMFM19], using A_∞ algebra structures imposed on $H(A)$ which are "adapted" to certain Massey products.

4. Final thoughts

We provide a few parting thoughts. We are not the only ones interested in further codifying the relationships between A_∞ algebras and Massey products as well as other structures: Burke [Bur15] expresses in a remark a desire also to relate A_∞ algebras to other

Golodity-adjacent structures, naming both homotopy Lie algebras and Eagon's resolution of the residue field of our local ring R. It appears that there may be multiple avenues by which DG algebra structures, properties and relationships may be able to be written in terms of A_∞ algebras, especially if its relationship to other peripheral topics are better understood.

Note that this same kind of understanding can be applied more generally to Golod modules, those for which (V.1) holds. The relationship between Golod modules and A_∞ algebras has been studied in [Bur15], also using their relationship to the bar construction under the hood. Though we use Golodity primarily as a vessel for understanding the relationship between A_∞ structures and Massey products, and as such our harvest regarding it is rather simple and already-tread, an inquiry into further Massey-product-like conditions, especially those regarding vanishing such as Golodity or formality albeit at a higher order, may be feasible for both rings and modules. Indeed, anywhere where triviality at certain orders regarding A_∞ algebra structures can be found, conditions with Massey products can likely be substituted for more concrete understanding using the template provided here.

One additional benefit of relating some of these universal-resolution-adjacent topics to A_∞ algebras is it may allow for more streamlined computational implementation of some of the relevant topics. A vast quantity of the topics here, as well as others in similar veins regarding Tate resolutions, Koszul duals and the like, already have implementations in Macaulay2 (see [Moo, `findTrivialMasseyOperation`], [LL21, `koszulDual`], [BBG$^+$, `dualRingToric`], and [ADE$^+$, `tateResolution`]), and A_∞ algebras are no exception [ES]. A greater understanding of the underlying A_∞ algebra structures may allow for future constructions to rely more closely on existing computational resources, as well as lead to the discovery for new uses of these structures in explicit and computational settings, especially elsewhere in this suite of universal resolutions.

VI

COHOMOLOGICAL SUPPORT VARIETY STRUCTURES FOR CERTAIN MONOMIAL IDEALS

In Section 1, we provide a synopsis of the relevant background information on cohomological support varieties and their calculation in the general and monomial settings, introducing also for the sake of resolving monomial ideals the Taylor resolution. In Section 2, we extrapolate on an existing construction of the cohomological support variety in the monomial setting, leveraging the fact that our resulting resolution is over $\overline{k}$ and exploiting the relationship between the Taylor resolution and cellular cochain complexes of simplicial complexes, whence Theorem C is derived.

1. More background on cohomological support varieties and monomial ideals

We provide a synopsis of the current landscape of the realizability problem for cohomological support varieties, a note on the implications of certain properties a cohomological support variety can take on, and some existing work on calculating cohomological support varieties of monomial ideals. This section draws largely from [BGP25].

1.1. Realizability and Consequences of Cohomological Support Varieties. Recent work has endeavored to investigate how much information about M and R is encoded in the cohomological support variety. This comprises two questions:

- What values can $\mathrm{V}_R(M)$ take on? (This is known as the *realizability question* for cohomological support varieties)
- What do those values tell us about M and R?

Theorem VI.1 ([Pol21, Theorem B])**.** If Q is a regular local ring, R is a complete intersection if and only if $\mathrm{V}_R(R) = \emptyset$.

Let $\mathrm{P}_k^R(t)$ denote the Poincaré series of M over R. By, for example, [Avr98, Proposition 3.3.2], every coefficient of the Poincaré series $\mathrm{P}_k^R(t)$ is bounded above by the corresponding coefficient of the power series

$$\frac{(1+t)^{\operatorname{edim} R}}{1-\sum_{j=1}^{\operatorname{codepth} R} \operatorname{rank}_k H_j(K^R)t^{j+1}},$$

where edim is the embedding dimension and K^R is the Koszul complex on a minimal generating set of $\mathfrak{m}R$. We say R is *Golod* if this equality holds at each entry for $M = k$.

Theorem VI.2 ([BGP24, Theorem D])**.** Let R be a Golod local ring. For any bounded complex M of finitely generated R-modules with $\mathrm{H}(M) = 0$, expressed as an E-module by assigning to each ξ_i the trivial action, the cohomological support variety $\mathrm{V}_R(M)$ is either all of $\mathbb{A}_k^n$ or a (conical) hypersurface, and every hypersurface is a cohomological support variety of some such complex. Furthermore, if R is not a hypersurface ring, then $\mathrm{V}_R(R) = \mathbb{A}_k^n$.

Henceforth in this chapter and the next, we will consider the two following cases simultaneously:

- $\widehat{R} = Q/I$ with Q a regular local ring and I in the square of the maximal ideal of Q (the *local case*),
- $R = Q/I$ where Q is a positively graded polynomial algebra over a field and I an ideal generated by homogeneous forms of degree at least 2 (the *graded case*).

In such cases we will call Q/I a *minimal regular presentation* of R.

Theorem VI.3 ([BGP25, Theorem 6.16])**.** Assume I is minimally generated by five monomials in Q. Then the cohomological support variety of R is either a coordinate subspace of $\mathbb{A}_k^5$ or a union of two hyperplanes. More precisely, up to reordering of the generators, $\mathrm{V}_R(R)$ is either $\mathscr{V}(\chi_1\chi_5)$ or a hyperplane subspace.

In this paper we continue the investigation of $\mathrm{V}_R(R)$ when R is a monomial ideal launched by [BGP25].

1.2. Constructing $\mathrm{V}_R(R)$ for R a monomial ideal. We provide a blueprint for constructing $\mathrm{V}_R(R)$ for R a monomial ideal, based primarily on [BGP25], rather than a comprehensive account on how the underlying structures can be used to constructing cohomological support varieties in a more general setting. Those interested in performing these constructions in a more general setting should consult [Pol19].

Our calculations of the cohomological support varieties of monomial ideals in this paper stem from the following

Proposition VI.4 ([BGP25, Proposition 2.8])**.** For an R-complex M with $H(M)$ finitely generated, fix a Koszul resolution F of M. Then

$$\mathrm{V}_R(M) = \{a \in \mathbb{A}^n_{\overline{k}} : H(\widehat{\mathscr{C}}_{E_a}(F)) \neq 0\} \cup \{0\} = \{a \in \mathbb{A}^n_{\overline{k}} : H_p(\widehat{\mathscr{C}}_{E_a}(F)) \neq 0\} \cup \{0\},$$

where $p \in \{\text{even}, \text{odd}\}$ and

$$\widehat{\mathscr{C}}_{E_a}(F) := \cdots \longrightarrow F_{\text{even}} \otimes_Q \overline{k} \xrightarrow{d_a} F_{\text{odd}} \otimes_Q \overline{k} \xrightarrow{d_a} F_{\text{even}} \otimes_Q \overline{k} \longrightarrow \cdots \tag{VI.1}$$

where

$$d_a := \partial^F \otimes 1 + \sum \xi_i \otimes a_i$$

and $\overline{k}$ is a fixed algebraic closure of k.

Letting B_{even} and B_{odd} be bases of $F_{\text{even}} \otimes_Q \overline{k}$ and $F_{\text{odd}} \otimes_Q \overline{k}$ respectively, the above describes d_a as an automorphism on $F \otimes_Q \overline{k}$ expressible as follows:

$$\begin{array}{c c} & \begin{array}{cc} B_{\text{even}} & B_{\text{odd}} \end{array} \\ \begin{array}{c} B_{\text{even}} \\ B_{\text{odd}} \end{array} & \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \end{array}.$$

This formulation leads us to the following

Corollary VI.5. Under the conditions above,

$$\mathrm{V}_R(M) = \{a \in \mathbb{A}^n_{\overline{k}} : H(\widehat{\mathscr{C}}'_{E_a}(F)) \neq 0\} \cup \{0\},$$

where $\widehat{\mathscr{C}}'_{E_a}(F)$ is the vector space $F \otimes_Q \overline{k}$ endowed with the automorphism given by d_a.

This proposition allows us to explicitly determine the cohomological support variety of an R-module M, given a sufficiently simple Koszul resolution F of M.

We now introduce some prerequisites specific to monomial ideals. To start, henceforth in this chapter and the next, let Q/I be a minimal regular presentation of R with residue or base field k, depending on whether we lie in the local case or the graded case, respectively. We will let d be the embedding dimension of Q and fix a regular sequence $x_1, \ldots, x_d$ of Q in the former case, and name our variables as such in the latter, so that the notion of a *monomial* of Q is well-defined. We let $\boldsymbol{f} = f_1, \ldots, f_n$ be a sequence of monomials in Q such that $I = (f_1, \ldots, f_n)$ is a proper ideal of Q.

Notation VI.6. Let $[n]$ denote the ordered set $\{1, \ldots, n\}$. If J is a set of integers, ordered or unordered, we let $|J|$ denote its cardinality and $\operatorname{sort}(J)$ denote the ordered set with the same elements as J, ordered from least to greatest. If J is ordered, $\operatorname{sgn}(J)$ denote the sign of the permutation of J yielding $\operatorname{sort}(J)$. If J and K are disjoint sets of integers, we let JK denote their concatenation.

The following construction can be traced back to [Tay66]:

Construction VI.7. Each f_j can be written as

$$f_j = x_1^{a_{j1}} \cdots x_d^{a_{jd}} \qquad \text{for some } a_{ji} \geqslant 0\,.$$

Given a subset J of $[n]$, set

$$f_J := x_1^{a_{J1}} \cdots x_d^{a_{Jd}} \quad \text{where } a_{Ji} = \max\{a_{ji} : j \in J\}\,.$$

In particular, $f_j = f_{\{j\}}$. The *Taylor complex on* $\boldsymbol{f}$ (with respect to $x_1, \ldots, x_d$) is a free Q-complex $T = T(\boldsymbol{f})$, defined as follows. As a free graded Q-module, $T^\natural$ can be assigned a basis in degree i given by

$$\{b'_J : J \subseteq [n] \text{ with } |J| = i\},$$

and if J is an ordered set we write $b'_J = \operatorname{sgn}(J) b'_{\operatorname{sort}(J)}$. The differential on T is the Q-linear map determined by

$$\partial(b'_J) = \sum_{i=1}^{s} (-1)^{i-1} \frac{f_J}{f_{J \setminus \{j_i\}}} b'_{J \setminus \{j_i\}} \quad \text{with } J = \{j_1 < \cdots < j_s\}.$$

We equip the Taylor complex with a Q-bilinear product, defined on basis elements by

$$b'_J \cdot b'_K = \operatorname{sgn}(JK) \frac{f_J f_K}{f_{J \cup K}} b'_{J \cup K}.$$

Note that

$$b'_J \cdot b'_K = 0 \quad \text{if } J \cap K \neq \emptyset.$$

Proposition VI.8 ([Tay66, Theorem 12])**.** $T(\boldsymbol{f})$ under these conditions (namely, when $\boldsymbol{f}$ is a sequence of monomials) is a DG Q-algebra resolution of R.

Consequently, we will refer to $T(\boldsymbol{f})$ as the *Taylor resolution of* R *via* $\boldsymbol{f}$, or simply the *Taylor resolution* of R when $\boldsymbol{f}$ is clear. A more general construction of the Taylor complex in which $\boldsymbol{f}$ is not necessarily a sequence of monomials and which is not necessarily a resolution can be found, for example, in [Yuz99].

Proposition VI.9. $T(\boldsymbol{f})$ is naturally an E-module by letting $\xi_i \times -$ be given by $b_i \times -$, and is as such a Koszul resolution of R.

As such, we are free to use $T(\boldsymbol{f})$ as our Koszul resolution F of R in our calculations of $\mathrm{V}_R(R)$.

Remark VI.10 (The Taylor resolution versus the Koszul complex)**.** The Taylor resolution is a modification of the Koszul complex in the monomial ideal case which ensures a resolution. In the Koszul complex, if we assign to each copy of Q the product of the corresponding generators of our ideal, then our differential maps "preserve" these assignments in the sense that

$$\text{assignment}_{\text{source}} \times \text{ coefficient of map} = \text{assignment}_{\text{target}}\,, \tag{VI.2}$$

allowing for a resolution when our generators form a regular sequence. We observe this same mantra when constructing the Taylor resolution, but in order to construct a resolution when our sequence may not be regular, our assignments of our copies of Q ought to correspond to the least common multiples (LCMs) of the corresponding generators of our ideal, rather than their product.

Note that our construction of $\mathrm{V}_R(R)$ only requires us to understand $T(\boldsymbol{f}) \otimes \overline{k}$. The structure of $T(\boldsymbol{f}) \otimes \overline{k}$ is much simpler than that of $T(\boldsymbol{f})$ alone:

Construction VI.11 ($\boldsymbol{T(f) \otimes \overline{k}}$)**.** Let a_{ji} be as in Construction VI.7. As a free graded $\overline{k}$-module, $(T(\boldsymbol{f}) \otimes \overline{k})^\natural$ can be assigned a basis in degree i given by

$$\{b_J : J \subseteq [n] \text{ with } |J| = i\}\,,$$

and if J is an ordered set we write $b_J = \operatorname{sgn}(J) b_{\operatorname{sort}(J)}$. The differential on $T(\boldsymbol{f}) \otimes \overline{k}$ is the $\overline{k}$-linear map determined on basis elements by

$$\partial(b_J) = \sum_{f_J = f_{J \setminus \{j_i\}}} (-1)^{i-1}\, b_{J \setminus \{j_i\}} \quad \text{where } J = \{j_1 < \cdots < j_s\}.$$

We equip $T(\boldsymbol{f}) \otimes \overline{k}$ with a $\overline{k}$-bilinear product, defined on basis elements by

$$b_J \cdot b_K = \begin{cases} \operatorname{sgn}(JK)\, b_{J \cup K} & \text{if } f_J f_K = f_{J \cup K} \text{ and } J \cap K = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

2. Decomposing $\boldsymbol{T(f) \otimes \overline{k}}$

2.1. Our decomposition. The complex $T(\boldsymbol{f}) \otimes \overline{k}$ can be naturally decomposed into subcomplexes. Throughout this section, we will use the following

Example VI.12. Let $Q = \mathbb{Q}[x_1, \ldots, x_7]$, $\boldsymbol{f} = (x_1x_2, x_2x_3, \ldots, x_6x_7, x_7x_1)$, and $J = \{1, 3, 5\}$.

Definition VI.13. Let

$$S_J = \{K \mid f_K = f_J\},$$

let $T_J(\boldsymbol{f})$ be the restriction of $T(\boldsymbol{f}) \otimes \overline{k}$ to the basis b_K with $K \in S_J$, and let $M_J = \bigcup_{K \in S_J} K$.

Lemma VI.14. $M_J = M_{J'}$ if and only if $f_J = f_{J'}$.

Proof. Say that $M_J = M_{J'}$. M_J is the union of a positive number of sets K such that $f_K = f_J$, so, since the LCM of a union of sets is the LCM of the LCMs of those sets, f_{M_J} is the LCM of many copies of f_J, and is thus itself f_J. Thus we also have $f_{J'} = f_{M'_J} = f_{M_J}$. Conversely, if $f_J = f_{J'}$, then $S_J = S_{J'}$, so $M_J = M_{J'}$. □

Lemma VI.15. $M_J = \{j \in [n] \mid f_j | f_J\}$.

Proof. If $f_j | f_J$, then $f_{J \cup \{j\}} = f_J$, so $j \in M_J$. If $j \in M_J$, then there exists some $K \ni j$ such that $f_j | f_K = f_J$. □

Remark VI.16. In Example VI.12, we have

$$f_J = x_1x_2x_3x_4x_5x_6,$$

$$S_J = \{\{1,3,5\}, \{1,3,4,5\}, \{1,2,4,5\}, \{1,2,3,5\}, \{1,2,3,4,5\}\}, \text{ and}$$

$$M_J = \{1,2,3,4,5\},$$

$$T_J(\boldsymbol{f}) = \underset{5}{\overline{k}} \xrightarrow[\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}]{} \underset{4}{\overline{k}^3} \xrightarrow[\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}]{} \underset{3}{\overline{k}},$$

where we assign the three non-trivial vector spaces in $T_J(\boldsymbol{f})$ bases

$$\left(b_{\{1,2,3,4,5\}}\right), \left(b_{\{1,3,4,5\}}, b_{\{1,2,4,5\}}, b_{\{1,2,3,5\}}\right), \left(b_{\{1,3,5\}}\right),$$

respectively.

Proposition VI.17. $T_J(\boldsymbol{f})$ is a subcomplex of $T(\boldsymbol{f}) \otimes \overline{k}$, and furthermore $T(\boldsymbol{f}) \otimes \overline{k}$ is the direct sum of $T_{M_J}(\boldsymbol{f})$ over all distinct M_J.

Proof. Recall that $T_{M_J}(\boldsymbol{f})$ is the restriction of $T(\boldsymbol{f}) \otimes \overline{k}$ to the basis b_K with $K \in S_J$. Note that for each $K \subseteq [n]$ there is exactly one set S_J containing K, since the set of distinct S_J is a partition of the sets $K \subseteq [n]$ by the values f_K. Thus it suffices to show that, if we consider the differential of $T(\boldsymbol{f}) \otimes \overline{k}$ as a matrix by using the basis $\{b_K \mid K \subset [n]\}$, then whenever $M_J \neq M_{J'}$, which by Lemma VI.14 is to say whenever $f_J = f_{J'}$, then the coefficient of this matrix from b_J to b_J is zero. This is clear from the description of the differential in Construction VI.11. □

We refer to the complexes $T_J(\boldsymbol{f})$ as *Taylor subcomplexes* on $\boldsymbol{f}$ (with respect to $(x_1, \ldots, x_d)$).

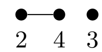
The differential structures of these subcomplexes are those of the cohomologies of corresponding simplicial complexes. Let us be precise. Let $K_\Delta = M_K \setminus K$ and let $\Delta_J = \{K_\Delta \mid K \in S_J\}$, noting that $K \in S_J$ if and only if $K_\Delta \in \Delta_J$ if and only if $M_J = M_K$.

Lemma VI.18. Δ_J is a simplicial complex.

Proof. We wish to show that if $K_\Delta \in \Delta_J$, then any subset of K_Δ is in Δ_J. This is equivalent to saying that if $f_K = f_J$, then for any $K \subset L \subset M_J$ we have $f_L = f_J$. Since $L \supset K$ we have $f_J = f_K | f_L$, and since $L \subset M_J$ we have $f_L | f_{M_J} = f_J$, which together suffice as the LCMs in question are all those of monomials. □

Consider the augmented chain complex $\widetilde{C}_\bullet(\Delta_J, \overline{k})$. Its i-th entry has basis the i-simplices K_Δ of Δ_J, interpreting the trivial simplex as having dimension -1. As such, the i-th entry of the augmented cochain complex $\widetilde{C}^\bullet(\Delta_J, \overline{k})$ is the dual $\operatorname{Hom}(\widetilde{C}_i(\Delta_J, \overline{k}), \overline{k})$, and thus we can assign it a basis consisting of the resulting duals of the i-simplices of Δ_J: namely, the dual $(K_\Delta)^\vee$ of an i-simplex K_Δ sends K_Δ to 1 and the other i-simplices of Δ_J to zero. We use c'_K to refer to $(K_\Delta)^\vee$. Thus, there is a basis of the i-th entry of $\widetilde{C}^\bullet\left(\Delta_J, \overline{k}\right)$ consisting of the terms c'_K over all K such that K_Δ is an i-simplex of Δ_J. We call Δ_J the *monomial subcomplex* of J.

Remark VI.19. In Example VI.12,

$$\Delta_J = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2,4\}\} =$$ 

The bases of the entries of $\widetilde{C}^{\bullet}(\Delta_J, \overline{k})$ are as follows:

$$\begin{aligned} \widetilde{C}^{-1}(\Delta_J, \overline{k}) &: (\emptyset^{\vee}) \\ \widetilde{C}^{0}(\Delta_J, \overline{k}) &: (\{2\}^{\vee}, \{3\}^{\vee}, \{4\}^{\vee}) \\ \widetilde{C}^{1}(\Delta_J, \overline{k}) &: (\{2,4\}^{\vee})\,, \end{aligned}$$

and with this basis, we have

$$\widetilde{C}^{\bullet}(\Delta_J, \overline{k}) = \underset{-1}{\overline{k}} \xrightarrow[\begin{bmatrix}1\\1\\1\end{bmatrix}]{} \underset{0}{\overline{k}^3} \xrightarrow[\begin{bmatrix}1 & 0 & -1\end{bmatrix}]{} \underset{1}{\overline{k}}\ .$$

If J is an ordered set we let $c'_J = \operatorname{sgn}(J) c'_{\operatorname{sort}(J)}$. Furthermore, let $\operatorname{ksgn}(J)$ (the *cochain complex sign* of J) be -1 to the power of the number of elements of J at even positions of $\operatorname{sort}(M_J)$ and let

$$\operatorname{ksgn}(j, J) = \operatorname{ksgn}(J)\operatorname{ksgn}(\{j\}J)$$

$$c_J = \operatorname{ksgn}(J) b_J,$$

Lemma VI.20. Fix some $J \in [n]$. If for all $i \in \mathbb{Z}$ we let $S_{J,i}$ be an ordered set consisting of the elements of S_J which have size i, and choose bases $(c_K)|_{K \in S_{J,i}}$ and $(c'_K)|_{K \in S_{J,i}}$ for $(T_J(\boldsymbol{f}))_i$ and $\widetilde{C}^{|M_J|-i+1}\left(\Delta_J, \overline{k}\right)$ respectively for all $i \in \mathbb{Z}$, then fixing any $i_0 \in \mathbb{Z}$ the matrices

$$\partial_{i_0}(T_J(\boldsymbol{f})) \quad \text{and} \quad \partial^{|M_J|-i_0-1}\left(\widetilde{C}^{\bullet}\left(\Delta_J, \overline{k}\right)\right)$$

with $\overline{k}$-entries are identical.

Note that we do not draw an isomorphism between $T_J(\boldsymbol{f})$ and $\Sigma^{|M_J|-1}\widetilde{C}^\bullet\left(\Delta_J, \overline{k}\right)$ directly, as such a shift would introduce additional unwanted signs arising from the use of the suspension operator. However, if we were to shift $\widetilde{C}^\bullet\left(\Delta_J, \overline{k}\right)$ by naïvely re-indexing its entries instead, we could construct a true isomorphism.

Proof of Lemma VI.20. Fix some $j \in J$. It suffices to show that, whenever J and $J \setminus \{j\}$ lie in S_J, that is, whenever $f_J = f_{J\setminus\{j\}}$, the coefficient from c_J to $c_{J\setminus\{j\}}$ is exactly that from c'_J to $c'_{J\setminus\{j\}}$. We begin with the first of these coefficients. The coefficient from b_J to $b_{J\setminus\{j\}}$ in $T_J(\boldsymbol{f})$ is $(-1)^{|\{p\in J \,|\, p<j\}|}$ by definition, so the coefficient from c_J to $c_{J\setminus\{j\}}$ is

$$\begin{aligned}(-1)^{|\{p\in J \,|\, p<j\}|}\operatorname{ksgn}(J)\operatorname{ksgn}(J\setminus\{j\}) &= (-1)^{|\{p\in J \,|\, p<j\}|}(-1)^{|\{p\in M_J | p<j\}|}\\ &= (-1)^{|\{p\in J_\Delta \,|\, p<j\}|}\\ &= (-1)^{|\{p\in (J\setminus\{j\})_\Delta \,|\, p<j\}|},\end{aligned}$$

which is by definition the coefficient from c'_J to $c'_{J\setminus\{j\}}$ in $\widetilde{C}^\bullet\left(\Delta_J, \overline{k}\right)$. $\square$

Remark VI.21. In Example VI.12, we have $\operatorname{ksgn}(\{1,2,3,5\}) = -1$: there is a single element of the set {1,2,3,5}, namely 2, which has even index in the ordered set $\operatorname{sort}(M_{\{1,2,3,5\}}) = (1,2,3,4,5)$. We have

$$\operatorname{ksgn}(\{1,3,5\}) = 1, \quad \operatorname{ksgn}(\{1,3,4,5\}) = -1, \quad \operatorname{ksgn}(\{1,2,4,5\}) = 1,$$
$$\operatorname{ksgn}(\{1,2,3,5\}) = -1, \quad \operatorname{ksgn}(\{1,2,3,4,5\}) = 1.$$

This is reflected in the matrices we already calculated for $T_J(\boldsymbol{f})$ and $\widetilde{C}^\bullet(\Delta_J, \overline{k})$: for some $K \in S_J, j \in K$, the matrix entry taking b_K to $b_{K\setminus\{j\}}$ is that taking c_K to $c_{K\setminus\{j\}}$ times $\operatorname{ksgn}(K)\operatorname{ksgn}(K\setminus\{j\})$.

Let us explicitly record the action of our DG multiplication structure on the basis $\{c_J\}$. When non-zero, we have

$$b_j c_J = \begin{cases} \operatorname{ksgn}(j, J) c_{\{j\}J} = \operatorname{ksgn}(j, J) \operatorname{sgn}(\{j\}J) c_{\{j\} \cup J} & \text{if } f_{\{j\}} f_J = f_{\{j\} \cup J}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $f_{\{j\}} f_J = f_{\{j\} \cup J}$, then $f_J \neq f_{\{j\} \cup J}$, which means that this multiplication will never have entries within one of our summands of $T(\boldsymbol{f})$.

2.2. A double complex structure on $\widehat{\mathscr{C}}'_{E_a}(F)$ for $\boldsymbol{f}$ equidegree. We begin with some further notation for concision. Let $a_J = \prod_{j \in J} a_j$ and define x_J similarly, where in the latter J may be a multiset. Second, we may omit $(\boldsymbol{f})$ from $T_J(\boldsymbol{f})$ when $\boldsymbol{f}$ is clear. Third, henceforth, we will let underlined strings of hex digits (or integers generally, when their delineation is clear) refer to ordered sets (e.g. $\underline{12A} := (1, 2, 10)$).

The differential d_a on $\widehat{\mathscr{C}}'_{E_a}(F)$, which is identified with $T(\boldsymbol{f}) \otimes_Q \overline{k}$, is the sum of the following:

- the maps $\partial T_{M_J}(\boldsymbol{f})$ over all distinct M_J, our *intra-Taylor-subcomplex maps*,
- and the maps

$$a_j b_j : T_{M_J}(\boldsymbol{f}) \to T_{\{j\} \cup M_J}(\boldsymbol{f}),$$
$$c_K \mapsto a_j b_j c_K = \operatorname{ksgn}(j, J) c_{\{j\}J}$$

  over all distinct j, M_J such that $\gcd(\boldsymbol{f}_j, \boldsymbol{f}_{M_J}) = 1$, our *inter-Taylor-subcomplex maps*.

Recall the following

Definition VI.22 (Taylor graph, [BGP25, Definition 6.3])**.** The *Taylor graph* associated with $\boldsymbol{f}$ has vertices the subsets of $[n]$ and directed edges from J to K for $J, K \subset [n]$ exactly when the coefficient of d_a taking b_J to b_K is non-zero.

Our existing examples have quite large Taylor graphs, so we use the smaller example from [BGP25, Example 6.10]:

Example VI.23 ([BGP25, Example 6.10])**.** Let $n = 4$ and

$$\boldsymbol{f} = (x_1x_2, x_2x_3, x_3x_4, x_4x_5).$$

The Taylor graph of $\boldsymbol{f}$ is as follows:

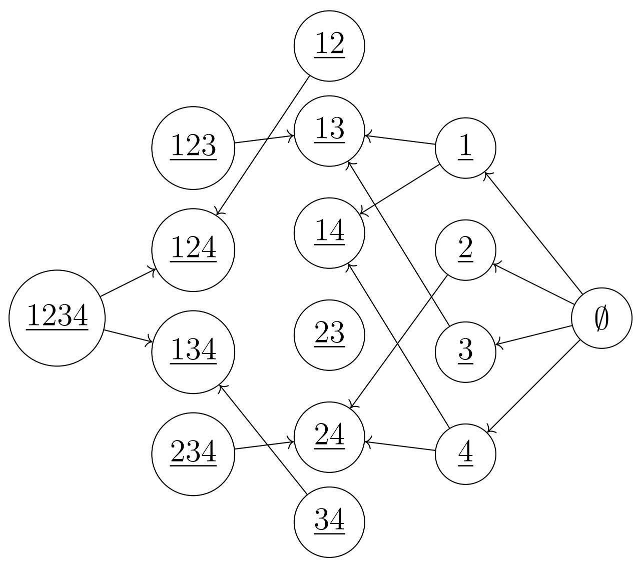


FIGURE 1. Taylor graph for $(x_1x_2, x_2x_3, x_3x_4, x_4x_5)$, cf. [BGP25, Figure 1]

Note that the non-zero coefficients of d_a taking some b_J to some b_K are each one of two types:

- Those for which J is the union of K and some element—those given by our intra-Taylor-subcomplex maps. If the vertices of the Taylor graph are placed in columns according to their entries in the Taylor complex, as above, these coefficients are those corresponding to right-facing arrows, which point one unit to the right.
- Those for which K is the union of J and some element. These are those given by our inter-Taylor-subcomplex maps. If the vertices of the Taylor graph are placed in columns according to their entries in the Taylor complex, as above, these coefficients are those corresponding to left-facing arrows, which point one unit to the left.

The subsequent work observes that if we can assign to each basis vector b_J an integer such that every edge in the Taylor graph has the integer assigned to the target one less than the edge assigned to the source, then we can construct a complex whose differential is given by d_a and whose i-th entry has basis those vectors assigned i. That is, if we can assign to our Taylor graph a *weak grading*, then we will be able to construct such a complex:

Definition VI.24. A *weak grading* (cf. [Zha15, Section 3], which provides a definition of a *weakly-graded* poset analogous to our definition here) of a graph is an assignment of an integer weight to each vertex such that each edge has the weight of its target one less than that of its source. We call such a graph *weakly gradable*.

Lemma VI.25. If our Taylor graph has a weak grading Σ, then d_a begets a chain complex differential on the graded vector space with basis $\{b_J \mid J \subset [n]\}$ and for which b_J is assigned the grade Σ_J given to it by Σ.

Letting B_i be a basis for the vector space assigned grade $i \in \mathbb{Z}$ under this lemma, our "chain complex differential" reveals nothing more than that d_a can be described as an automorphism on $\widehat{\mathscr{C}}'_{E_a}(T(\boldsymbol{f}))$ expressible as follows (thanks to our choice of partition B_i):

$$\begin{array}{c} \\ \vdots \\ B_2 \\ B_1 \\ B_0 \\ B_{-1} \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \cdots & B_2 & B_1 & B_0 & B_{-1} & \cdots \end{array} \\ \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & * & 0 & 0 & 0 & \cdots \\ \cdots & 0 & * & 0 & 0 & \cdots \\ \cdots & 0 & 0 & * & 0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}.$$

Example VI.26. Continuing Example VI.23, we can weakly grade our Taylor graph as in Figure 2, assigning to each subset of $[n]$ the integer heading the column in which it lies.

Finding a weak grading of a Taylor graph is not necessarily simple, so we will construct a new, simpler graph through which a weak grading of our Taylor graph can be built:

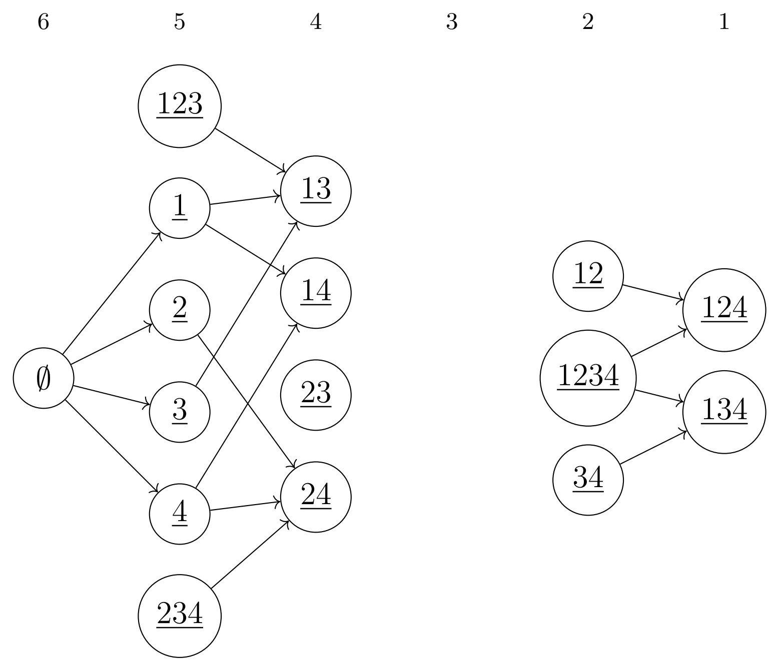


FIGURE 2. Weak grading of the Taylor graph for $(x_1x_2, x_2x_3, x_3x_4, x_4x_5)$, cf. [BGP25, Figure 1]

Definition VI.27. For some set of monomial generators of an ideal, partition the vertices of its Taylor graph by LCM: that is, for each $J \subseteq [n]$, place the vertex corresponding to J in the subset containing M_J. The *Taylor subcomplex graph* is the quotient of this Taylor graph by this partition. We use $T_{M_J}(\boldsymbol{f})$ to label the vertex corresponding to the subset containing M_J (so that it shares a name with the Taylor subcomplex to which it corresponds), though we may occasionally refer to this vertex as simply T_J.

Lemma VI.28. If our Taylor subcomplex graph has a weak grading Σ, then d_a begets a chain complex differential on the graded vector space with basis $\{b_J \mid J \subset [n]\}$ and for which b_J is assigned the grade $|J| + 2\Sigma_{T_J}$.

Proof. By Lemma VI.25 it suffices to show that $J \mapsto |J| + 2\Sigma_{T_J}$ is a weak grading of our Taylor graph. Consider an edge from J to J' in our Taylor graph. If it is given by an intra-Taylor-subcomplex map, then $|K| = |K'| + 1$ and $\Sigma_{T_J} = \Sigma_{T_{J'}}$. If it is given by an inter-Taylor-subcomplex map, then $|K| = |K'| - 1$ and $\Sigma_{T_J} = \Sigma_{T_{J'}} + 1$. In either case we have

$$|J'| + 2\Sigma_{T_{J'}} = |J| + 2\Sigma_J - 1,$$

which is to say that $J \mapsto |J| + 2\Sigma_{T_J}$ is a weak grading of our Taylor graph, completing our proof. □

We call the chain complex resulting from this lemma $T^{\Sigma}(\boldsymbol{f}, a)$.

Example VI.29. Continuing Example VI.23, our partition of vertices preceding our Taylor subcomplex graph would be as in Figure 3, which would yield the Taylor subcomplex graph in Figure 4, to which we have assigned a weak grading. The weak grading of the Taylor graph implied via Lemma VI.28 by the weak grading of our Taylor subcomplex graph given in Figure 4 is that shown in Figure 2.

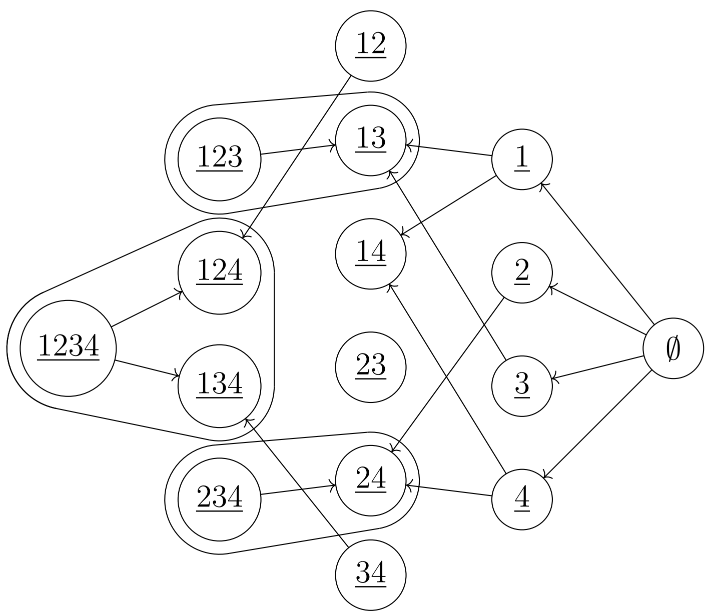


FIGURE 3. Taylor graph for $(x_1x_2, x_2x_3, x_3x_4, x_4x_5)$ with a partition by Taylor subcomplex, cf. [BGP25, Figure 1]

Theorem VI.30. If our Taylor subcomplex graph has a weak grading Σ,

$$\mathrm{V}_R(R) = \{a \in \mathbb{A}^n_{\overline{k}} : H(T^{\Sigma}(\boldsymbol{f}, a)) \neq 0\} \cup \{0\}\,.$$

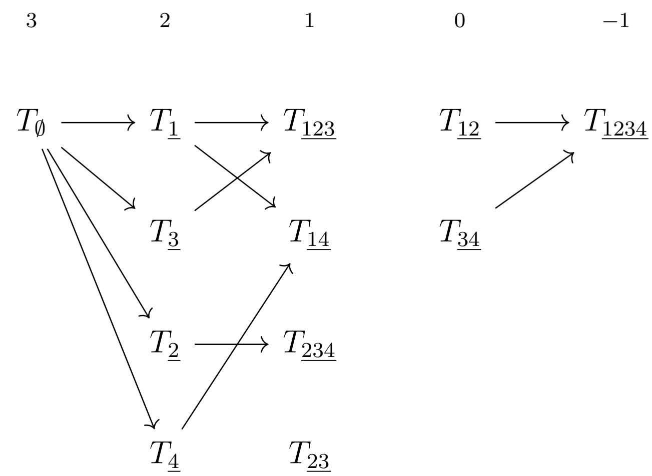


FIGURE 4. Taylor subcomplex graph for $(x_1x_2, x_2x_3, x_3x_4, x_4x_5)$ with a weak grading

Under this condition, using the basis $\{b_J \mid J \subseteq [n]\}$, the linear maps in $T^\Sigma(\boldsymbol{f}, a)$ can be given collectively by assigning matrices with entries polynomials over $a_1, \ldots, a_n$ to each integer, so that selecting some set of values for each a_i yields the differential maps of the corresponding chain complex $T^\Sigma(\boldsymbol{f}, a)$.

Proof. This characterization of $\mathrm{V}_R(F)$ is clear by Corollary VI.5 as $\widehat{\mathscr{C}}'_{E_a}(T(\boldsymbol{f}))$ and $T^\Sigma(\boldsymbol{f}, a)$ are identical as vector spaces with automorphisms, and our matrix assignment is clear from the definition of d_a. □

Example VI.31. Let $d = 6$, and let $\boldsymbol{f} = (x_{\underline{12}}, x_{\underline{34}}, x_{\underline{56}}, x_{\underline{135}}, x_{\underline{246}})$. We wish to draw the Taylor subcomplex graph of this sequence of monomials. Omitting the Taylor subcomplexes

$$T_{\underline{14}}, T_{\underline{15}}, T_{\underline{24}}, T_{\underline{25}}, T_{\underline{34}}, T_{\underline{35}}, T_{\underline{124}}, T_{\underline{125}}, T_{\underline{134}}, T_{\underline{135}}, T_{\underline{234}}, T_{\underline{235}}$$

from our graph for cleanliness (as they would all be isolated vertices), the remainder of our Taylor subcomplex graph, along with an attempt to weakly grade it, is in Figure 5. The graph there is not weakly gradable: this can be seen by considering the Taylor subcomplexes $T_\emptyset$, $T_{\underline{1}}$, $T_{\underline{12}}$, $T_{\underline{4}}$, and $T_{\underline{12345}}$.

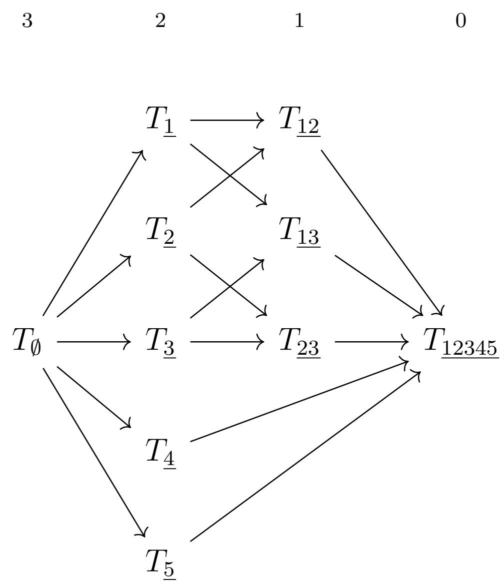


FIGURE 5. An attempt to weakly grade the Taylor subcomplex graph of $(x_{12}, x_{34}, x_{56}, x_{135}, x_{246})$

In order to take advantage of our understanding of weak gradings, we prove their existence when $\boldsymbol{f}$ is equidegree:

Lemma VI.32. Whenever $\boldsymbol{f}$ is equidegree, its Taylor subcomplex graph admits a weak grading.

Proof. Let $\Sigma_{T_J} = -\left\lfloor \frac{\deg f_J}{\deg f_1} \right\rfloor$. If $f_{\{j\}\cup J} = f_{\{j\}} f_J$, which is to say that there is an edge in our Taylor subcomplex graph from T_J to $T_{\{j\}\cup J}$, then

$$\begin{aligned}\Sigma_{T_{\{j\}\cup J}} &= -\left\lfloor \frac{\deg f_{\{j\}\cup J}}{\deg f_1} \right\rfloor = -\left\lfloor \frac{\deg f_{\{j\}\cup J}}{\deg f_1} \right\rfloor = -\left\lfloor \frac{\deg \left(f_{\{j\}} f_J\right)}{\deg f_1} \right\rfloor \\ &= -\left\lfloor \frac{\deg f_{\{j\}} + \deg f_J}{\deg f_1} \right\rfloor = -\left\lfloor \frac{\deg \left(f_{\{j\}} f_J\right)}{\deg f_1} \right\rfloor = -\left\lfloor \frac{\deg f_{\{j\}}}{\deg f_1} + \frac{\deg f_J}{\deg f_1} \right\rfloor,\end{aligned}$$

so by our assumption that $\boldsymbol{f}$ is equidegree, $\deg f_{\{j\}} = \deg f_1$ so $\Sigma_{T_{\{j\}\cup J}} = \Sigma_{T_J} - 1$, completing our proof. □

Corollary VI.33. Every cohomological support variety of a ring with a minimal regular presentation given by an equigenerated monomial ideal with n generators is the set of points in $\mathbb{A}_k^n$ such that a chain complex of vector spaces with total dimension 2^n with entries defined by polynomials in the n variables begetting our affine space has non-trivial homology.

Proof. The chain complex $T^{\Sigma}(\boldsymbol{f}, a)$ given in Theorem VI.30 fits the description of the chain complex in the statement of the claim, and when $\boldsymbol{f}$ is equigenerated our Taylor subcomplex graph admits a weak grading by Theorem VI.32. $\square$

The Macaulay2 method which currently computes cohomological support varieties ([GLP, `extKoszul`]) does so by calculating the homology of d_a on $\widehat{\mathscr{C}}'_{E_a}(F)$ for F some minimal resolution. Since the Taylor resolution is not necessarily minimal, and we make no explicit claims regarding the distribution of our chain complex, we cannot make claims that our calculations of cohomological support varieties will always consider strictly smaller matrices. However, whenever for each generator of a monomial ideal there is some variable such that the power of that variable is largest at that generator, for example in the ideal

$$(x_1^2 x_2, x_2^2 x_3, \ldots, x_d^2 x_1) \tag{VI.3}$$

over $\mathbb{Q}[x_1, \ldots, x_d]$ for $d \geq 2$ a positive integer, the Taylor resolution is minimal [Ale17, Theorem 4.4]. This would require the existing method to calculate the homology of a 2^d-by-2^d square matrix, though with some further implementation it could instead calculate kernels and images of 2^{d-1}-by-2^{d-1} square matrices by leveraging the decomposition of $F \otimes_Q \overline{k}$ into $F_{\text{even}} \otimes_Q \overline{k}$ and $F_{\text{odd}} \otimes_Q \overline{k}$ as in Proposition VI.4. For our calculations, the consideration of a 2^{d-1}-by-2^{d-1} matrix would be the worst case, necessitating that our Taylor graph was concentrated in two consecutive grades. For some perspective on this worst case, the ideal (VI.3), using the construction afforded by Theorem VI.30 and Theorem VI.32, will yield non-zero entries in all grades from 0 to $d - 2\lfloor 2d/3 \rfloor$.

We can impose even further structure on our chain complexes by construction. We follow the convention in which double complexes' arrows go down and to the right, with the first entry decreasing by one when following a right-facing arrow, and with the second entry decreasing by one when following a down-facing arrow:

Theorem VI.34. If our Taylor subcomplex graph has a weak grading Σ, then d_a begets a double complex differential on the bigraded vector space with basis $\{b_J \mid J \subseteq [n]\}$ which assigns b_J bigrade $(|J| + \Sigma_{T_J}, \Sigma_{T_J})$, such that the vertical differential is the sum of our inter-Taylor-subcomplex maps and the horizontal differential is the sum of our intra-Taylor-subcomplex maps. Furthermore, the row with grade i is the direct sum of chain complexes identical to $T_J(\boldsymbol{f})$ for $\Sigma_{T_J} = i$ except that the grade of each basis element is decreased by Σ_{T_J}. Further still, the naïve totalization of this double complex (given by taking the sum of the vertical and horizontal differentials, without imposing any further signs) is $T^{\Sigma}(\boldsymbol{f}, a)$.

Proof. Note that any coefficient of d_a born of an intra-Taylor-subcomplex map is of bidegree $(1, 0)$ under this bigrading, and any coefficient of d_a born of an inter-Taylor-subcomplex map is of bidegree $(0, -1)$ under this bigrading. To show that this is a double complex, one must show that

$$d_{v,a}^2, d_{h,a}^2, d_{v,a}d_{h,a} + d_{h,a}, d_{v,a} = 0.$$

Note that for any basis vector b_J, the images of these three maps are linear sums of disjoint sets of basis vectors, so it suffices to note that their sum, $(d_{v,a} + d_{h,a})^2$, is zero. However, $d_{v,a} + d_{h,a}$ is simply d_a, which we already know has square zero.

We quickly address the claim regarding the decomposition of each row into familiar chain complexes. Since we can see that the basis vectors corresponding to the chain complexes $T_J(\boldsymbol{f})$ in question have the correct bigrades, it suffices to show that there are no horizontal differentials between entries not corresponding to subsets sharing an LCM. However, we have already stated that our horizontal differential is a sum only of intra-Taylor-subcomplex maps, which implies this quite clearly. □

Example VI.35. Continuing with Example VI.23, the weak grading given in Figure 4 yields the double complex Figure 6, in which each non-zero entry is given by the collection of vertices whose corresponding basis vectors generate it.

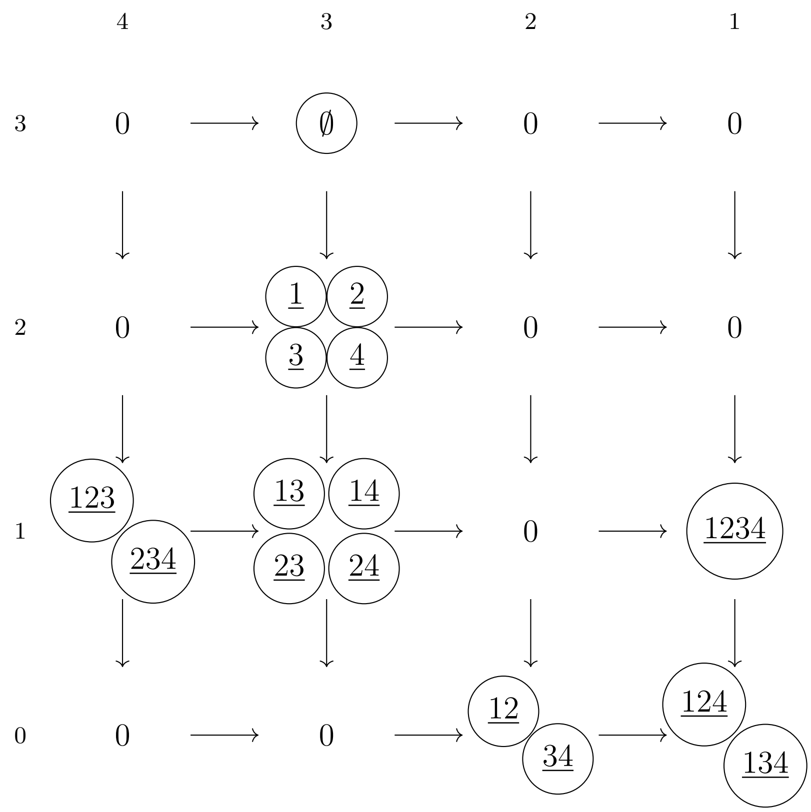


FIGURE 6. Double complex corresponding to the weak grading of the Taylor subcomplex graph of $(x_1x_2, x_2x_3, x_3x_4, x_4x_5)$ given by Figure 4, in which each non-zero entry is given by the collection of vertices whose corresponding basis vectors generate it, cf. [BGP25, Figure 1]

We will use $T^{\Sigma}_{M_J}(\boldsymbol{f}, a)$ to refer to the row-direct-summand corresponding to $T_{M_J}(\boldsymbol{f})$ mentioned above, though we may drop the $\boldsymbol{f}$ and a from this notation when they are clear or replace M_J with simply J. We let c^{Σ}_J refer to the basis element of this double complex afforded by this corollary corresponding to c_J.

Corollary VI.36. If the Taylor subcomplex graph of $\boldsymbol{f}$ is weakly gradable, $T^{\Sigma}(\boldsymbol{f}, a)$ can be decomposed by partitioning the basis $\{b_J \mid J \subseteq [n]\}$ by the connected component of the Taylor subcomplex graph containing T_J. This decomposition also applies to the double complex given in Theorem VI.34.

Finally, we provide information regarding whether a weak grading exists for some given sequence of monomials:

Theorem VI.37. The Taylor subcomplex graph of a sequence $\boldsymbol{f}$ admits a weak grading if and only if there are no two subsets $S, T \subset [n]$ such that any pair of elements of S or pair of elements of T are coprime, $f_S = f_T$, and $|S| \neq |T|$.

Proof. If two such subsets exist then a Taylor subcomplex graph is not weakly gradable: in any Taylor subcomplex graph we can get from $T_\emptyset$ to T_{M_S} via either $|S|$ or $|T|$ edges.

If no two such subsets exist, we will iteratively assign weights to our T_J for increasing sizes of $J = M_J$ by the procedure which follows. If T_J is not connected to any subcomplex which has already been assigned a weight, assign it a weight of zero. Note that in this case, a new connected component of the induced subgraph G of Taylor subcomplex graph consisting of those subcomplexes already considered is created. If T_J is connected to some subcomplexes which have already been assigned a weight, note that the edges between T_J and these all have T_J as their target. If these source subcomplexes share an assigned weight whenever they share a connected component of G, then we can adjust our weights by connected component to allow for a compatible assignment of weight to T_J. This leaves us the case in which there are two subcomplexes T_{J_1} and T_{J_2} in the same connected component of G which have been assigned different weights and both of which have an edge directed towards T_J.

Consider now this case. Consider a path p from T_{J_1} to T_{J_2}, possibly moving backwards along edges. We claim that such a path p' exists which moves backwards along some non-negative number of edges and then moves forwards along some non-negative number of edges. To prove this, it suffices to show that, if at any point in p we move forwards along one edge to some and then backwards along another, going from g_1 to g_2 to g_3 for some g_1, g_2, g_3, we could have instead moved backwards along an edge first and then forwards along another and arrived at the same destination. If our edges correspond to the elements $f_{\underline{i}}$ and $f_{\underline{j}}$ respectively, then for some $K \subset [n] \setminus \{i, j\}$ we have

$$g_1 = \{j\} \sqcup K, \qquad g_2 = \{i, j\} \sqcup K, \qquad g_3 = \{i\} \sqcup K,$$

where $f_{\underline{i}}$ and $f_{\underline{j}}$ are coprime with $f_{\underline{j}K}$ and $f_{\underline{i}K}$ respectively. This implies that $f_{\underline{i}}$ and $f_{\underline{j}}$ are coprime with f_K, so we can replace g_2 with K in p, so a path p' as described above exists.

There must be some unique element T_L along p' with the highest assigned weight. Then we have paths moving forwards along edges from T_L to both T_{J_1} and T_{J_2} in G. Our iterative assignment of weights to the vertices of G and the fact that in this assignment T_{J_1} and T_{J_2} have different weights tell us that these paths have different lengths. By considering the additional edges from these to T_J, we have two paths of different lengths from T_L to T_J moving forward along edges. Say that these edges correspond to sets S and T. Then we have

$$\gcd(f_L, f_S) = 1, \quad \gcd(f_L, f_T) = 1, \quad \text{and} \quad f_{L\cup S} = f_{L\cup T}.$$

The first two of these indicate that

$$f_{L\cup S} = f_L f_S \quad \text{and} \quad f_{L\cup T} = f_L f_T,$$

yielding $f_S = f_T$, so S and T satisfy the conditions given in the theorem statement, contradiction. □

This theorem tells us that it is relatively easy to check whether a Taylor subcomplex admits a weak grading: a non-weakly-gradable Taylor subcomplex graph will have two paths from $T_\emptyset$ to some other vertex with different lengths, and furthermore these paths will each be given by coprime sets of elements of $\boldsymbol{f}$.

VII

COMPUTING COHOMOLOGICAL SUPPORT VARIETY FOR CERTAIN MONOMIAL IDEALS

In Section 1, we will perform some up-front labor regarding cohomological support varieties of edge ideals in particular. In Section 2, we will use our construction begetting Theorem C to manually prove Theorem D. In Section 3 we describe the code used to prove Computation E and Computation F. In Section 4 we discuss potential next steps, as well as provide some calculations which may be useful for future calculations of cohomological support varieties of edge ideals.

1. Simplicial complexes concerning edge ideals

We have a decent understanding of the simplicial complexes Δ_J corresponding to edge ideals thanks to [Koz99], whose results we will recall here in a more relevant form. Consider first $\Delta_{\underline{12\ldots i}}$ for some $2 \leq i < n-1$. $\Delta_{\underline{12\ldots i}}$ has $i-2$ points, indexed from 2 to $i-1$, and consists of subsets of this set of indices containing no two adjacent values.

Lemma VII.1 ([Koz99, Proposition 4.6])**.** Say $i \geq 1$ and $\boldsymbol{f}$ is the edge ideal of a cycle with at least $i+2$ generators. If $3 \mid i$, then $\Delta_{\underline{1\ldots i}}$ has trivial reduced homology. Otherwise, its reduced homology is $\mathbb{Z}$ at entry $\left\lfloor \frac{n}{3} - 1 \right\rfloor$ and zero elsewhere.

Now consider some arbitrary M_S which is not $M_{[n]}$. Note that M_S can be considered as a union of sets of consecutive values (considering 1 and $4m+2$ consecutive) which is as small as possible. Then Δ_{M_S} will be isomorphic to the set union of the corresponding $\Delta_{\underline{1\ldots i}}$ complexes above, and will have homology their direct sum, whence their reduced homology will be able to be determined.

Lemma VII.2 ([Koz99, Proposition 5.2])**.** Say $n \geq 2$ and $\boldsymbol{f}$ is the edge ideal of a cycle with n generators. Then the reduced homology of $\Delta_{\underline{1\ldots n}}$ is zero except

$$\begin{cases} \mathbb{Z}^2 \text{ at entry } \frac{n}{3}-1 & \text{when } n \equiv 0 \pmod 3, \\ \mathbb{Z} \text{ at entry } \frac{n-1}{3}-1 & \text{when } n \equiv 1 \pmod 3, \\ \mathbb{Z} \text{ at entry } \frac{n-2}{3} & \text{when } n \equiv 2 \pmod 3. \end{cases}$$

In both of these cases, by the universal coefficient theorem, the reduced cohomologies and reduced homologies of these simplicial complexes will be isomorphic. Additionally, before continuing, it is worth noting that [Koz99] contains information regarding the generating simplices of these simplicial complexes, which, should this path of inquiry prove fruitful, may be worth investigating further, especially given that we are working primarily with cohomology.

We also wish to elaborate a bit on the calculation and simplification of ksgn. Note that $\operatorname{ksgn}(S)$ seeks the number of elements of S which have even indices in $\operatorname{sort}(M_S \setminus S)$. We note that this is the sign of the permutation which takes the reversed $\operatorname{sort}(S)$, which we will write as $\operatorname{rsort}(S)$, and concatenates $\operatorname{sort}(M_S \setminus S)$ to it:

$$\operatorname{ksgn}(S) = \operatorname{sgn}(\operatorname{rsort}(S)\operatorname{sort}(M_S \setminus S)).$$

This will ease some of our sign-calculation pains moving forward.

Finally, as theorized in private communication with E. Grifo, we have the following lemma which encapsulates a lower bound on our support varieties which will meaningfully simplify the remainder of our calculations:

Lemma VII.3. Any edge ideal of size $4m+2$ has in its support variety that given by $\mathscr{V}(a_{\underline{13\ldots(4m+1)}} + a_{\underline{24\ldots(4m+2)}})$.

Proof. For $i \in \mathbb{Z}$, let

- $S_i^+ = \sigma^i\left(\{i \in [4m+2] \mid i \in 1, 2 \pmod 4\}\right)$,

- $S_i^\ell = \sigma^i\left(S \setminus \{1\}\right)$,
- $S_i^r = \sigma^i\left(S \setminus \{4m+2\}\right)$, and
- $S_i^- = \sigma^i\left(S \setminus \{4m+1, 4m+2\}\right)$.

Consider the subspace of $T^\Sigma(\boldsymbol{f}, a)$ spanned by the set of basis elements

$$\bigcup \langle \sigma^2 \rangle \{b_{\mathrm{sort}(S^\ell)}, b_{\mathrm{sort}(S^r)}\}$$

where σ is the permutation $(1\ 2 \ldots 4m+2)$. Its intersection with $d_a T^\Sigma(\boldsymbol{f}, a)$ is zero, and its image under d_a in $T^\Sigma(\boldsymbol{f}, a)$ lies in the subspace with basis

$$\langle \sigma^2 \rangle b_{S^+} \cup \langle \sigma^2 \rangle b_{S^-}.$$

These two subspaces have dimension $4m+2$, so it suffices to show that the square matrix between these given by d_a has zero determinant when $a_{\underline{13\ldots(4m+1)}} + a_{\underline{24\ldots(4m+2)}} = 0$. We will let $a_{i(4m+2)+j}$ refer to a_j for T an ordered set and $i, j \in \mathbb{Z}$. We have

$$d_a\left(b_{S_i^-}\right) = a_{i-1} b_{S_i^r} + a_{i-2} b_{S_{i-4}^\ell} \qquad \text{and} \qquad d_a\left(b_{S_i^+}\right) = -b_{S_i^r} + b_{S_i^\ell},$$

so the map between these given by d_a is given by the matrix

$$\begin{array}{c} \\ b_{S^r} \\ b_{S^\ell} \\ b_{S_4^r} \\ b_{S_4^\ell} \\ b_{S_8^r} \\ \\ b_{S_{-4}^\ell} \end{array} \begin{array}{c} \begin{array}{ccccccc} b_{S^-} & b_{S^+} & b_{S_4^-} & b_{S_4^+} & b_{S_8^-} & & b_{S_{-4}^+} \end{array} \\ \begin{bmatrix} a_{4m+1} & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & a_6 & \cdots & 0 \\ 0 & 0 & 0 & 0 & a_7 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{4m} & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

which has determinant $a_{\underline{13\ldots4m+1}} + a_{\underline{24\ldots4m+2}}$, completing our proof. □

2. The edge ideals of 6- and 10-cycles

The edge ideal of a d-cycle is $\boldsymbol{f} = x_1x_2, x_2x_3, \ldots, x_dx_1$ in $k[x_1, \ldots, x_d]$.

2.1. The case $d = 6$. Let $d = 6$, let $\boldsymbol{f} = x_{\underline{12}}, x_{\underline{23}}, \ldots, x_{\underline{61}}$, and let σ be the permutation $(1\ 2 \ldots 6)$. Our sets M_J are:

- $\emptyset$,
- $\langle\sigma\rangle\underline{1}$, of which there are 6,
- $\langle\sigma\rangle\underline{12}$, of which there are 6,
- $\langle\sigma\rangle\underline{123}$, of which there are 6,
- $\langle\sigma\rangle\underline{1234}$, of which there are 6,
- $\underline{123456}$, of which there is 1,
- $\langle\sigma^2\rangle\underline{14}$, of which there are 3.

A quotiented version of our weakly gradable Taylor subcomplex graph, which we have quotiented by a partition such that the graph induced by any two subsets of our partition is either empty or a unidirectional complete bipartite graph on the two subsets, is as follows:

$$\begin{array}{ccccccc} & & & & \langle\sigma\rangle T_{\underline{123}} & & \\ & & & \nearrow & & \searrow & \\ T_{\emptyset} & \longrightarrow & \langle\sigma\rangle T_{\underline{1}} & & & & T_{\underline{123456}} \\ & & & \searrow & & & \\ & & & & \langle\sigma^2\rangle T_{\underline{14}} & & \end{array}$$

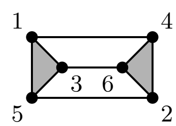
$$\langle\sigma\rangle T_{\underline{12}} \longrightarrow \langle\sigma\rangle T_{\underline{1234}}$$

This graph has two connected components, which by Corollary VI.36 we may treat separately. We consider first the component with five entries. We get the following double complex from Theorem VI.34:

$$\begin{array}{c} T_{\emptyset}^{\Sigma} \\ \downarrow \\ \bigoplus\langle\sigma\rangle T_{\underline{1}}^{\Sigma} \\ \downarrow \\ \bigoplus\langle\sigma\rangle T_{\underline{123}}^{\Sigma} \oplus \bigoplus\langle\sigma^2\rangle T_{\underline{14}}^{\Sigma} \\ \downarrow \\ T_{\underline{123456}}^{\Sigma} \end{array}$$

Given this double complex, we can determine if its totalization has any homology using spectral sequences. Here we exploit the relationship between our T_J^{Σ} complexes and reduced cellular cochain complexes. Let us first calculate our Δ_J so we can better understand their homology:

- $\Delta_{\emptyset}$, each of $\langle\sigma\rangle\Delta_{\underline{1}}$, and each of $\langle\sigma^2\rangle\Delta_{\underline{14}}$ are $\emptyset$ — no other f_J are the same as those from these subsets of $\boldsymbol{f}$.
- $\Delta_{\underline{123}}$ is the point 2, which we can write $\underset{2}{\bullet}$. $\langle\sigma\rangle$ can be applied to this to get the others.
- $\Delta_{\underline{123456}}$ consists of all subsets of $\underline{123456}$ without adjacent pairs (including the pair $\underline{16}$), which looks like the following:



In each of these cases, the reduced homology will be free, so our reduced cohomologies will be isomorphic by the universal coefficient theorem:

- $\emptyset$ has $\mathrm{H}^{-1}(\emptyset) \cong \mathbb{Z}$ and no other cohomology,
- $\underset{2}{\bullet}$ has no reduced cohomology,
- $\Delta_{\underline{123456}}$ has $\mathrm{H}^1(\Delta_{\underline{123456}}) \cong \mathbb{Z}^2$ and no other cohomology.

We begin by taking horizontal derivatives. Letting H^i_J denote the reduced i-th cohomology of Δ_J, our E_1 page will be isomorphic to the following, where our isomorphisms from H^{i-1}_J to $(\mathrm{H}(T_J^{\Sigma}))_{|M_J|-\Sigma_{T_J}-i}$ are inherited from the map taking c'_J to c_J^{Σ}, where all non-zero entries are shown:

$$\begin{array}{ccc}
0 & \longrightarrow & \mathrm{H}_{\emptyset}^{-1} \\
\big\downarrow & & \big\downarrow \\
0 & \longrightarrow & \bigoplus \langle \sigma \rangle \, \mathrm{H}_{\underline{1}}^{-1} \\
\big\downarrow & & \big\downarrow \\
0 & \longrightarrow & \bigoplus \langle \sigma^2 \rangle \, \mathrm{H}_{\underline{14}}^{-1} \\
\big\downarrow & & \big\downarrow \\
\mathrm{H}^1_{\underline{123456}} & \longrightarrow & 0
\end{array}$$

Next we construct E_2. We fix a basis for each of our non-zero terms in order to define maps between our entries as matrices:

- $\mathrm{H}_{\emptyset}^{-1}$ is generated by $c'_{\emptyset}$,
- $\langle \sigma \rangle \, \mathrm{H}_{\underline{1}}^{-1}$ are generated by $\langle \sigma \rangle c'_{\underline{1}}$,
- $\langle \sigma^2 \rangle \, \mathrm{H}_{\underline{14}}$ are generated by $\langle \sigma^2 \rangle c'_{\underline{14}}$,
- $\mathrm{H}^1_{\underline{123456}}$ is generated by $c'_{\underline{2356}}$ and $-c'_{\underline{1346}}$.

Note that this includes generators such as $c'_{\underline{52}} = -c'_{\underline{25}}$. We can now define our E_1 maps by recalling that when non-zero, the value of $b_j c_J$ is

$$\operatorname{ksgn}(j, J) \operatorname{sgn}(\underline{j}J) c_{\operatorname{sort}(\underline{j}J)},$$

and our vertical maps are directly inherited from these multiplication maps:

$$\mathrm{H}_{\emptyset}^{-1} \to \bigoplus \langle \sigma \rangle \, \mathrm{H}_{\underline{1}}^{-1} : \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}, \qquad \bigoplus \langle \sigma \rangle \, \mathrm{H}_{\underline{1}}^{-1} \to \bigoplus \langle \sigma \rangle \, \mathrm{H}_{\underline{14}}^{-1} : \begin{bmatrix} a_4 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & a_5 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & a_6 & 0 & 0 & -a_3 \end{bmatrix}.$$

The first of these maps is injective. The second map is surjective so long as no i satisfies $a_i = a_{i+3} = 0$, contradicting our assumption provided in Section 1. Our E_2 page is isomorphic to the following, where all non-zero entries are shown:

$$\begin{array}{ccc}
0 & \longrightarrow & \ker \begin{bmatrix} a_4 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & a_5 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & a_6 & 0 & 0 & -a_3 \end{bmatrix} / \operatorname{im} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\mathrm{H}^1_{\underline{123456}} & \longrightarrow & 0
\end{array}$$

The top-right entry is generated by $a_1 c^{\Sigma}_{\underline{1}} + a_4 c^{\Sigma}_{\underline{4}}$ and $a_2 c^{\Sigma}_{\underline{2}} + a_5 c^{\Sigma}_{\underline{5}}$, and thus are left only to determine the images of these in the entry isomorphic to $\mathrm{H}^1_{\underline{123456}}$ under the E_2 map.

The images of these values under the vertical map are

$$-a_{\underline{13}} c^{\Sigma}_{\underline{13}} + a_{\underline{15}} c^{\Sigma}_{\underline{15}} + a_{\underline{24}} c^{\Sigma}_{\underline{24}} - a_{\underline{46}} c^{\Sigma}_{\underline{46}} \qquad \text{and} \qquad - a_{\underline{24}} c^{\Sigma}_{\underline{24}} + a_{\underline{26}} c^{\Sigma}_{\underline{26}} + a_{\underline{35}} c^{\Sigma}_{\underline{35}} - a_{\underline{15}} c^{\Sigma}_{\underline{15}}.$$

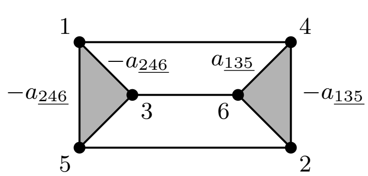
Noting that the c'_J corresponding to these c^{Σ}_J are all points, we can pull these back to achieve

$$-a_{\underline{13}} c^{\Sigma}_{\underline{123}} + a_{\underline{15}} c^{\Sigma}_{\underline{156}} + a_{\underline{24}} c^{\Sigma}_{\underline{234}} - a_{\underline{46}} c^{\Sigma}_{\underline{456}} \qquad \text{and} \qquad - a_{\underline{24}} c^{\Sigma}_{\underline{234}} + a_{\underline{26}} c^{\Sigma}_{\underline{126}} + a_{\underline{35}} c^{\Sigma}_{\underline{345}} - a_{\underline{15}} c^{\Sigma}_{\underline{156}}.$$

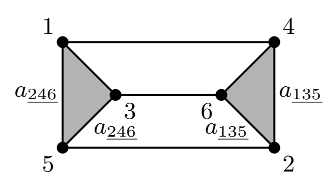
Finally, we take the image of these under our vertical map one more time, yielding

$$a_{\underline{135}} c^{\Sigma}_{\underline{1235}} - a_{\underline{135}} c^{\Sigma}_{\underline{1356}} - a_{\underline{246}} c^{\Sigma}_{\underline{2346}} - a_{\underline{246}} c^{\Sigma}_{\underline{2456}} \qquad \text{and} \qquad a_{\underline{246}} c^{\Sigma}_{\underline{2346}} + a_{\underline{246}} c^{\Sigma}_{\underline{1246}} + a_{\underline{135}} c^{\Sigma}_{\underline{1345}} + a_{\underline{135}} c^{\Sigma}_{\underline{1356}}.$$

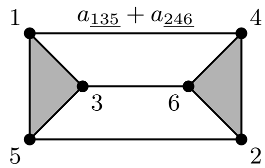
Under our isomorphism with our reduced cellular cochain complex, we can write this as the following weighted sum of segments in $\Delta_{\underline{123456}}$:



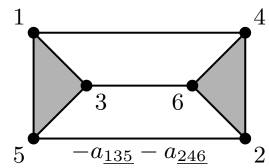
and



These are the following, up to a cocycle:



and



Thus the image of the chosen basis of our source E_2 entry under this E_2 map are $a_{\underline{135}} + a_{\underline{246}}$ times the basis of our target E_2 entry. Thus the subcomplex of $T^{\Sigma}(\boldsymbol{f}, a)$ corresponding to this component has trivial homology by our assumption that $a_{\underline{135}} + a_{\underline{246}} \neq 0$, yielding the exactness of this subcomplex of $T^{\Sigma}(\boldsymbol{f}, a)$.

Now we must consider the other connected component of our given weakly gradable Taylor subcomplex graph. It yields the double complex

$$\begin{array}{c} \bigoplus \langle \sigma \rangle T^{\Sigma}_{\underline{12}} \\ \downarrow \\ \bigoplus \langle \sigma \rangle T^{\Sigma}_{\underline{1234}} \end{array}$$

Again we determine our Δ_J and their reduced cohomologies:

- Each of $\langle \sigma \rangle \Delta_{\underline{12}}$ is $\emptyset$, which has only $\mathrm{H}^{-1} = \mathbb{Z}$,
- Each of $\langle \sigma \rangle \Delta_{\underline{1234}}$ are, respectively, $\langle \sigma \rangle \ \underset{2}{\bullet} \ \underset{3}{\bullet}$, which have $\mathrm{H}^0 = \mathbb{Z}$ and no other cohomology.

Our E_1 page is isomorphic to the following:

$$\begin{array}{c} \bigoplus \langle \sigma \rangle \ \mathrm{H}^{-1}_{\underline{12}} \\ \downarrow \\ \bigoplus \langle \sigma \rangle \ \mathrm{H}^{0}_{\underline{1234}} \end{array}$$

We fix a basis for our non-zero terms:

- $\langle \sigma \rangle \ \mathrm{H}^{-1}_{\underline{12}}$ are generated respectively by $\langle \sigma \rangle c'_{\underline{12}}$,
- $\langle \sigma \rangle \ \mathrm{H}^{-1}_{\underline{1234}}$ are generated respectively by $\langle \sigma \rangle c'_{\underline{124}}$.

Under these bases the vertical map is given by the matrix

$$\begin{bmatrix} -a_4 & 0 & -a_1 & 0 & 0 & 0 \\ 0 & -a_5 & 0 & -a_2 & 0 & 0 \\ 0 & 0 & -a_6 & 0 & -a_3 & 0 \\ 0 & 0 & 0 & a_1 & 0 & a_4 \\ -a_5 & 0 & 0 & 0 & -a_2 & 0 \\ 0 & a_6 & 0 & 0 & 0 & a_3 \end{bmatrix}$$

which up to a change of basis is

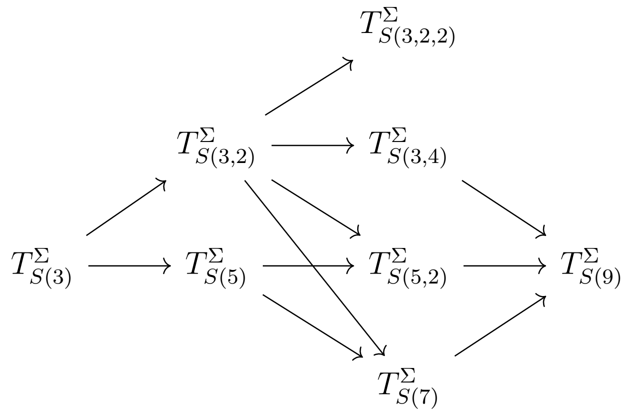
$$\begin{bmatrix} a_4 & a_1 & 0 \\ 0 & a_6 & a_3 \\ a_5 & 0 & a_2 \end{bmatrix} \oplus \begin{bmatrix} a_5 & a_2 & 0 \\ 0 & a_1 & a_4 \\ a_6 & 0 & a_3 \end{bmatrix}$$

both summands of which have determinant $a_{\underline{135}} + a_{\underline{246}}$. We have successfully proven that the cohomological support variety of this edge ideal is $\mathscr{V}(a_{\underline{135}} + a_{\underline{246}})$.

2.2. The case $d = 10$. Let $d = 10$ and let $f = (x_{\underline{12}}, \ldots, x_{\underline{A1}})$. Furthermore, let k have characterstic not equal to 2 or 5. We again know that our Taylor subcomplex graph will have two components, corresponding to the parity of the degrees of the LCMs of the sets of elements. We will start with the component yielding odd degrees. Our sets M_J are as follows, where in parentheses we have given our collections of sets names based on the lengths of sets of consecutive variables in their corresponding f_J values, where σ is the permutation $(1\ 2 \ldots 10)$:

- $\langle\sigma\rangle\{\underline{12}\}$, of which there are 10 ($S(3)$),
- $\langle\sigma\rangle\{\underline{125}, \underline{126}, \underline{127}, \underline{128}\}$, of which there are 40 ($S(3,2)$),
- $\bigcup\langle\sigma\rangle\{\underline{1234}\}$, of which there are 10 ($S(5)$),
- $\bigcup\langle\sigma\rangle\{\underline{1258}\}$, of which there are 10 ($S(3,2,2)$),
- $\bigcup\langle\sigma\rangle\{\underline{12567}, \underline{12678}\}$, of which there are 20 ($S(3,4)$),
- $\bigcup\langle\sigma\rangle\{\underline{12347}, \underline{12348}\}$, of which there are 20 ($S(5,2)$),
- $\bigcup\langle\sigma\rangle\{\underline{123456}\}$, of which there are 10 ($S(7)$), and
- $\bigcup\langle\sigma\rangle\{\underline{123456789}\}$, of which there are 10 ($S(9)$).

Our weakly gradable Taylor subcomplex graph quotiented by the above partition is as follows:



We then have the following double complex:

$$\begin{array}{c}
\bigoplus T^{\Sigma}_{S(3)} \\
\downarrow \\
\bigoplus T^{\Sigma}_{S(3,2)} \oplus \bigoplus T^{\Sigma}_{S(5)} \\
\downarrow \\
\bigoplus T^{\Sigma}_{S(3,2,2)} \oplus \bigoplus T^{\Sigma}_{S(3,4)} \oplus \bigoplus T^{\Sigma}_{S(5,2)} \oplus \bigoplus T^{\Sigma}_{S(7)} \\
\downarrow \\
\bigoplus T^{\Sigma}_{S(9)}
\end{array}$$

Now we take the horizontal homology using Lemma VII.1, yielding the following diagram, where all of the non-zero entries are shown:

$$\begin{array}{ccc}
0 & \longrightarrow & \bigoplus \mathrm{H}^{-1}_{S(3)} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \begin{array}{c} \bigoplus \mathrm{H}^{-1}_{S(3,2)} \\ \oplus \bigoplus \mathrm{H}^{0}_{S(5)} \end{array} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \begin{array}{c} \bigoplus \mathrm{H}^{-1}_{S(3,2,2)} \\ \oplus \bigoplus \mathrm{H}^{0}_{S(5,2)} \end{array} \\
\downarrow & & \downarrow \\
\bigoplus \mathrm{H}^{1}_{S(9)} & \longrightarrow & 0
\end{array}$$

Now in the right-hand column, since all of our simplicial complexes are quite simple, we can see that the first non-trivial map is injective and the second is surjective, so the homology at the middle non-trivial entry in that column has rank 10. It suffices to manually find distinct entries here which we can push down, then pull back, then push down, and show that the results yield non-trivial cocycles in $\bigoplus \mathrm{H}^1_{S(9)}$. These will prove the rank of our only potentially non-trivial E_2 map and consequently prove the exactness of our double complex. Consider

$$a_{\underline{125}}c_{\underline{125}} + a_{\underline{128}}c_{\underline{128}} - a_{\underline{248}}c_{\underline{348}} - a_{\underline{159}}c_{\underline{59A}}$$

Pushing this down yields

$$\begin{aligned} & - a_{\underline{1257}}c_{\underline{1257}} + a_{\underline{1258}}c_{\underline{1258}} + a_{\underline{1259}}c_{\underline{1259}} \\ & \quad - a_{\underline{1248}}c_{\underline{1248}} - a_{\underline{1258}}c_{\underline{1258}} + a_{\underline{1268}}c_{\underline{1268}} \\ & \quad - a_{\underline{1248}}c_{\underline{1348}} - a_{\underline{2468}}c_{\underline{3468}} + a_{\underline{248A}}c_{\underline{348A}} \\ & \quad + a_{\underline{1259}}c_{\underline{259A}} - a_{\underline{1359}}c_{\underline{359A}} + a_{\underline{1579}}c_{\underline{579A}} \end{aligned}$$

$$\begin{aligned} = & - a_{\underline{1257}}c_{\underline{1257}} + a_{\underline{1259}}(c_{\underline{1259}} + c_{\underline{259A}}) \\ & \quad - a_{\underline{1248}}(c_{\underline{1248}} + c_{\underline{1348}}) + a_{\underline{1268}}c_{\underline{1268}} \\ & \quad - a_{\underline{2468}}c_{\underline{3468}} + a_{\underline{248A}}c_{\underline{348A}} \\ & \quad - a_{\underline{1359}}c_{\underline{359A}} + a_{\underline{1579}}c_{\underline{579A}}. \end{aligned}$$

Pulling this back, which can be done manually using simplicial complexes by considering the pullbacks of the corresponding c' terms, yields

$$\begin{aligned} & - a_{\underline{1257}}c_{\underline{12567}} + a_{\underline{1259}}c_{\underline{1259A}} \\ & \quad - a_{\underline{1248}}c_{\underline{12348}} + a_{\underline{1268}}c_{\underline{12678}} \\ & \quad + a_{\underline{2468}}c_{\underline{34678}} + a_{\underline{248A}}c_{\underline{3489A}} \end{aligned}$$

$$- a_{\underline{1359}}c_{\underline{3459A}} + a_{\underline{1579}}c_{\underline{5679A}},$$

and pushing it down once more yields

$$\begin{aligned} & a_{\underline{12579}}c_{\underline{125679}} - a_{\underline{12579}}c_{\underline{12579A}} \\ & \quad - a_{\underline{12468}}c_{\underline{123468}} + a_{\underline{12468}}c_{\underline{124678}} \\ & \quad + (a_{\underline{12468}}c_{\underline{134678}} + a_{\underline{2468A}}c_{\underline{34678A}}) + a_{\underline{2468A}}c_{\underline{34689A}} \\ & \quad + a_{\underline{13579}}c_{\underline{34579A}} + (-a_{\underline{12579}}c_{\underline{25679A}} + a_{\underline{13579}}c_{\underline{35679A}}) \end{aligned}$$

$$\begin{aligned} & a_{\underline{12579}}(c_{\underline{125679}} - c_{\underline{12579A}} - c_{\underline{25679A}}) \\ & \quad + a_{\underline{12468}}(-c_{\underline{123468}} + c_{\underline{124678}} + c_{\underline{134678}}) \\ & \quad + a_{\underline{2468A}}c_{\underline{34678A}} + a_{\underline{2468A}}c_{\underline{34689A}} \\ & \quad + a_{\underline{13579}}c_{\underline{34579A}} + a_{\underline{13579}}c_{\underline{35679A}} \end{aligned}$$

By considering simplicial complexes, the first two lines clearly comprise trivial cocycles, whereas the last comprises $a_{\underline{13579}} + a_{\underline{2468A}}$ times a non-trivial cocycle, proving that this subcomplex is exact as we have assumed that this value is non-zero.

Now we must consider the even-degree LCMs. We have the following M_J, similarly partitioned:

- $\emptyset$ (1),
- $\langle\sigma\rangle\underline{1}$ ($S(2)$, 10),
- $\langle\sigma\rangle\{\underline{14}, \underline{15}, \underline{16}\}$ ($S(2,2)$, 25),
- $\langle\sigma\rangle\underline{123}$ ($S(4)$, 10),
- $\langle\sigma\rangle\underline{147}$ ($S(2,2,2)$, 10),
- $\langle\sigma\rangle\{\underline{1456}, \underline{1567}, \underline{1678}\}$ ($S(2,4)$, 30),
- $\langle\sigma\rangle\{\underline{1256}, \underline{1267}\}$ ($S(3,3)$, 15),
- $\langle\sigma\rangle\underline{12345}$ ($S(6)$, 10),

- $\langle\sigma\rangle\underline{145678}$ $(S(2,6),\ 10)$,
- $\langle\sigma\rangle\underline{125678}$ $(S(3,5),\ 10)$,
- $\langle\sigma\rangle\underline{123678}$ $(S(4,4),\ 5)$,
- $\langle\sigma\rangle\underline{1234567}$ $(S(8),\ 10)$,
- $\langle\sigma\rangle\underline{123456789A}$ $(S(10),\ 1)$.

This gives us the following:

$$\begin{array}{c}
T^{\Sigma}_{\emptyset} \\
\downarrow \\
\bigoplus T^{\Sigma}_{S(2)} \\
\downarrow \\
\bigoplus T^{\Sigma}_{S(2,2)} \oplus \bigoplus T^{\Sigma}_{S(4)} \\
\downarrow \\
\bigoplus T^{\Sigma}_{S(2,2,2)} \oplus \bigoplus T^{\Sigma}_{S(2,4)} \oplus \bigoplus T^{\Sigma}_{S(3,3)} \oplus \bigoplus T^{\Sigma}_{S(6)} \\
\downarrow \\
\bigoplus T^{\Sigma}_{S(2,6)} \oplus \bigoplus T^{\Sigma}_{S(3,5)} \oplus \bigoplus T^{\Sigma}_{S(4,4)} \oplus \bigoplus T^{\Sigma}_{S(8)} \\
\downarrow \\
T^{\Sigma}_{\underline{123456789A}}
\end{array}$$

Now we take the horizontal homology using Lemma VII.1 and Lemma VII.2, where all of the non-zero entries are shown:

$$
\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^{-1}_{\emptyset} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \bigoplus \mathrm{H}^{-1}_{S(2)} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \bigoplus \mathrm{H}^{-1}_{S(2,2)} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \begin{array}{c} \bigoplus \mathrm{H}^{-1}_{S(3,3)} \\ \oplus \bigoplus \mathrm{H}^{0}_{S(6)} \end{array} & \longrightarrow & \bigoplus \mathrm{H}^{-1}_{S(2,2,2)} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \begin{array}{c} \bigoplus \mathrm{H}^{0}_{S(2,6)} \\ \oplus \bigoplus \mathrm{H}^{0}_{S(3,5)} \\ \oplus \bigoplus \mathrm{H}^{1}_{S(8)} \end{array} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^{2}_{\underline{123456789A}} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

We can calculate the ranks of these maps. The top-right vertical map is injective. The map below can be seen to require the ratio of the above map in a source element in order to have image zero, so it has rank 9. The following map is surjective, as seen by considering images of $\langle\sigma\rangle c_{\underline{15}}$, so it has rank 10. The first non-zero vertical map in the middle column can be considered as follows. The images of $\bigoplus \mathrm{H}^0_{S(6)}$ biject via restriction to $\mathrm{H}^0_{S(2,6)}$. The images of $\bigoplus \mathrm{H}^{-1}_{\langle\sigma\rangle\{1267\}}$ inject via restriction to $\bigoplus \mathrm{H}^1_{S(8)}$. Thus it suffices to consider the map from $\bigoplus \mathrm{H}^{-1}_{\langle\sigma\rangle\{1256\}}$ to $\bigoplus \mathrm{H}^1_{S(3,5)}$. Since we may assume that $a_{\underline{13579}} + a_{\underline{2468A}} \neq 0$, this map has full rank. To summarize, we have the following ranks of maps on our E_1 page:

$$
\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^{-1}_{\emptyset} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\
0 & \longrightarrow & 0 & \longrightarrow & \bigoplus \mathrm{H}^{-1}_{S(2)} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 9} \\
0 & \longrightarrow & 0 & \longrightarrow & \bigoplus \mathrm{H}^{-1}_{S(2,2)} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 10} \\
0 & \longrightarrow & \begin{array}{c}\bigoplus \mathrm{H}^{-1}_{S(3,3)} \\ \oplus \bigoplus \mathrm{H}^{0}_{S(6)}\end{array} & \longrightarrow & \bigoplus \mathrm{H}^{-1}_{S(2,2,2)} \\
\downarrow & & \downarrow{\scriptstyle 25} & & \downarrow \\
0 & \longrightarrow & \begin{array}{c}\bigoplus \mathrm{H}^{0}_{S(2,6)} \\ \oplus \bigoplus \mathrm{H}^{0}_{S(3,5)} \\ \oplus \bigoplus \mathrm{H}^{1}_{S(8)}\end{array} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^{2}_{\underline{123456789A}} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

Thus we have the following dimensions on our E_2 page:

$$
\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 6 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 5 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

The subquotient of $\bigoplus \mathrm{H}^1_{S(2,2)}$ in our E_2 page can be generated by $\langle\sigma\rangle c_{\underline{16}}$ and $\sum\langle\sigma\rangle a_{\underline{15}}c_{\underline{15}} - \sum\langle\sigma^2\rangle a_{\underline{14}}c_{\underline{14}}$. It suffices to show that the first five of these can be taken by pushing and pulling onto a basis of our rank-5 entry in the middle column, and that the last of these can be taken by pushing and pulling onto some non-zero entry in the first column.

Let us first consider the image of $c_{\underline{16}}$. This will, up to a sign, by symmetry, give us the images of $\langle\sigma\rangle c_{\underline{16}}$. We are fortunate that signs need not be tracked in this calculation. Our first push down yields

$$a_3 b_3 c_{\underline{16}} + a_4 b_4 c_{\underline{16}} a_8 + b_8 c_{\underline{16}} + a_9 b_9 c_{\underline{16}}$$
$$= \pm a_3 c_{\underline{136}} \pm a_4 c_{\underline{146}} \pm a_8 c_{\underline{168}} \pm a_9 c_{\underline{169}}.$$

To pull these left, it is easier to consider simplicial complexes. Each of these c_S terms has a corresponding c'_S in a simplicial complex consisting of a single point, thus they can be pulled left by removing the point, so pulling left yields

$$\pm a_3 c_{\underline{1236}} \pm a_4 c_{\underline{1456}} \pm a_8 c_{\underline{1678}} \pm a_9 c_{\underline{169A}}.$$

To push down for the second time, we will record only the terms coming from $S(2,6)$, which will be sufficient to determine that we have found a non-trivial cocycle. These are

$$\pm a_{\underline{39}} b_9 c_{\underline{1236}} \pm a_{\underline{48}} b_8 c_{\underline{1456}} \pm a_{\underline{48}} b_4 c_{\underline{1678}} \pm a_{\underline{39}} b_3 c_{\underline{169A}}$$
$$= \pm a_{\underline{39}} c_{\underline{12369}} \pm a_{\underline{48}} c_{\underline{14568}} \pm a_{\underline{48}} c_{\underline{14678}} \pm a_{\underline{39}} c_{\underline{1369A}}.$$

Now it suffices to show that any cocycle with these coefficients from $S(2,6)$ is non-trivial in E_2, that is, that it can't be pulled back. Again, this is clear by considering simplicial complexes. Unless $a_{\underline{48}} = a_{\underline{39}} = 0$, contradicting our assumption that $a_{\underline{13579}} + a_{\underline{2468A}} \neq 0$, our corresponding c'_S terms consist of some non-zero formal sum of points among $\Delta_{\underline{12369A}}$ and $\Delta_{\underline{145678}}$, which is a sum of either one or two non-zero scalar multiples of points on at least one of these. Since both of these simplicial complexes have 3 points, this can not be pulled back. By the same logic, no formal sum of elements of $\langle\sigma\rangle c_{\underline{16}}$ can be pulled back along this arrow. Thus this space bijects onto its target space in E_2.

This leaves us to prove that our remaining basis vector of this space in E_2, $\sum\langle\sigma\rangle a_{\underline{15}}c_{\underline{15}} - \sum\langle\sigma^2\rangle a_{\underline{14}}c_{\underline{14}}$, has a nontrivial image in E_3. We push down for the first time:

$$\begin{aligned}
&\sum\langle\sigma\rangle\left(a_{\underline{715}}b_7c_{\underline{15}} + a_{\underline{815}}b_8c_{\underline{15}} + a_{\underline{915}}b_9c_{\underline{15}}\right)\\
&\qquad - \sum\langle\sigma^2\rangle\left(a_{\underline{714}}b_7c_{\underline{14}} + a_{\underline{814}}b_8c_{\underline{14}}\right)\\
&= \sum\langle\sigma\rangle\left(\mathrm{ksgn}(7,\underline{15})a_{\underline{715}}c_{\underline{715}} + \mathrm{ksgn}(8,\underline{15})a_{\underline{815}}c_{\underline{815}} + \mathrm{ksgn}(9,\underline{15})a_{\underline{915}}c_{\underline{915}}\right)\\
&\qquad - \sum\langle\sigma^2\rangle\left(\mathrm{ksgn}(7,\underline{14})a_{\underline{714}}c_{\underline{714}} + \mathrm{ksgn}(8,\underline{14})a_{\underline{814}}c_{\underline{814}}\right)\\
&= \sum\langle\sigma\rangle\left(\mathrm{ksgn}(7,\underline{15})a_{\underline{715}}c_{\underline{715}} + \mathrm{ksgn}(8,\underline{15})a_{\underline{815}}c_{\underline{815}} + \mathrm{ksgn}(9,\underline{15})a_{\underline{915}}c_{\underline{915}}\right)\\
&\qquad - \sum\langle\sigma\rangle\left(\mathrm{ksgn}(1,\underline{58})a_{\underline{815}}c_{\underline{815}}\right)
\end{aligned}$$

Noting that $\sigma^i\,\mathrm{ksgn}(8,\underline{15}) = \mathrm{ksgn}(1,\underline{58}) = 1$ for all i, we are left with

$$\sum\langle\sigma\rangle\,\mathrm{ksgn}(7,\underline{15})a_{\underline{715}}c_{\underline{715}} + \mathrm{ksgn}(9,\underline{15})a_{\underline{915}}c_{\underline{915}}.$$

We pull back for the first time using simplicial complexes:

$$\sum\langle\sigma\rangle\,\mathrm{sgn}(7156)a_{\underline{157}}c_{\mathrm{sort}(\underline{1567})} + \mathrm{sgn}(915A)a_{\underline{159}}c_{\mathrm{sort}(\underline{159A})}.$$

We replace our second term with a shift:

$$\begin{aligned}
&\sum\langle\sigma\rangle\,\mathrm{sgn}(7156)a_{\underline{157}}c_{\mathrm{sort}(\underline{1567})} + \mathrm{sgn}(5716)a_{\underline{715}}c_{\mathrm{sort}(\underline{1567})}\\
&= 2\sum\langle\sigma\rangle\,\mathrm{sgn}(7156)a_{\underline{157}}c_{\mathrm{sort}(\underline{1567})}
\end{aligned}$$

Note that $\sigma^i\,\mathrm{sgn}(\underline{715}) = 1$ for all i. We push down for the second time:

$$\begin{aligned}
&2\sum\langle\sigma\rangle\,\mathrm{sgn}(37156\,\mathrm{sort}(24))a_{\underline{1357}}c_{\mathrm{sort}(13567)}\\
&\qquad + \mathrm{sgn}(97156\,\mathrm{sort}(8A))a_{\underline{1579}}c_{\mathrm{sort}(15679)}
\end{aligned}$$

We can shift the second term for future simplicity:

$$2\sum\langle\sigma\rangle\,\mathrm{sgn}(37156\,\mathrm{sort}(24))a_{\underline{1357}}c_{\mathrm{sort}(13567)}$$

$$+\operatorname{sgn}(53712\operatorname{sort}(46))a_{\underline{1357}}c_{\operatorname{sort}(12357)}.$$

We pull back for the second time:

$$\begin{aligned}&2\sum\langle\sigma\rangle\operatorname{sgn}(3715624)a_{\underline{1357}}\left(c_{\operatorname{sort}(\underline{123567})}-c_{\operatorname{sort}(\underline{134567})}\right)\\&\quad+\operatorname{sgn}(5371246)a_{\underline{1357}}\left(c_{\operatorname{sort}(\underline{123457})}-c_{\operatorname{sort}(\underline{123567})}\right)\\&=2\sum\langle\sigma\rangle\operatorname{sgn}(3715624)a_{\underline{1357}}\left(2c_{\operatorname{sort}(\underline{123567})}-c_{\operatorname{sort}(\underline{134567})}-c_{\operatorname{sort}(\underline{123457})}\right).\end{aligned}$$

We push down for the third and final time:

$$\begin{aligned}&\sum\langle\sigma\rangle a_{\underline{13579}}\big(\operatorname{sgn}(9371562)2c_{\operatorname{sort}(\underline{1235679})}\\&\qquad+\operatorname{sgn}(9371564)c_{\operatorname{sort}(\underline{1345679})}\\&\qquad+\operatorname{sgn}(9371542)c_{\operatorname{sort}(\underline{1234579})}\big)\\&=\sum\langle\sigma\rangle a_{\underline{13579}}\big(2c_{\operatorname{sort}(\underline{1235679})}\\&\qquad-c_{\operatorname{sort}(\underline{1345679})}-c_{\operatorname{sort}(\underline{1234579})}\big).\end{aligned}$$

We can shift the last term:

$$2\sum\langle\sigma\rangle a_{\underline{13579}}\left(c_{\operatorname{sort}(\underline{1235679})}-c_{\operatorname{sort}(\underline{1345679})}\right).$$

Now consider $\Delta_{\underline{1\ldots A}}$. We wish to show that here,

$$2\sum\langle\sigma\rangle a_{\underline{13579}}\left(c_{\operatorname{sort}(\underline{1235679})}-c_{\operatorname{sort}(\underline{1345679})}\right)$$

has non-trivial cohomology. Let $v_{\operatorname{sort}(S)}$ be the coefficent of the simplex corresponding to S in $\Delta_{1\ldots A}$ in a proposed preimage of the above cocycle and let $v_S=\operatorname{sgn}(S)v_{\operatorname{sort}(S)}$. Note that this cocycle has dot product $10(a_{\underline{13579}}+a_{\underline{2468A}})$ with the following formal sum of simplices:

$$\sum\langle\sigma\rangle\left(\underline{246}-2\cdot\underline{247}+\underline{258}\right).$$

This yields the following conditions imposed by our v_S terms:

$$10(a_{\underline{13579}} + a_{\underline{2468A}}) = \sum \langle \sigma \rangle \left((2v_{\underline{24}} - v_{\underline{26}}) - 2\left(v_{\underline{24}} - 2v_{\underline{25}}\right) + 2\left(v_{\underline{25}} + v_{\underline{26}}\right) \right) = 0,$$

contradiction.

3. Computational implementations

We have posted to `https://github.com/mgintz289/levels` a fork of the Macaulay2 package `ThickSubcategories.m2`. It introduces the method `equigeneratedMonomialCSV` and a test file `test-gintz-thesis.m2`. The method

- takes an equidegree set of monomial ideals $\boldsymbol{f}$ as input,
- decomposes the double complex described by partitioning its basis by the degrees of the LCMs of its entries modulo the degree of the entries of $\boldsymbol{f}$,
- computes the support of the homology of each component, and
- returns their union.

This code builds our double complex using the values of Σ_J using Corollary VI.33. The integration of our Σ_J terms into this workflow should make it easy to rework this code to generate cohomological support varieties for other monomial ideals which can be constructed using this framework.

The test file verifies Computation VII.4 and Computation VII.5:

Computation VII.4. The edge ideal of a 14-cycle over $\mathbb{Q}$ has support variety

$$\mathscr{V}(a_1a_3 \cdots a_{13} + a_2a_4 \cdots a_{14}).$$

Computation VII.5. The cohomological support varieties of ideals in polynomial rings over $\mathbb{Q}$ with minimal generating sets with 6 equidegree monomials are all one of the following up to order:

- a linear subspace,
- a union of two hyperplanes,

- $\mathscr{V}(a_{\underline{135}} + a_{\underline{246}})$.

Of course, we have made no statements regarding non-equigenerated monomial ideals or other base fields, so we have not proven the original problem posed in [BGP25].

This code constructs a collection of equigenerated monomial ideals such that every cohomological support variety of such an ideal, provided its minimal generating set has 6 generators, is represented, possibly excluding $\{0\}$ (though, in practice, this is not the case). Our construction revolves around the following

Lemma VII.6. The cohomological support variety of a monomial ideal can be determined by whether, for any given $J \subseteq [n]$ and $J \not\ni i \in [n]$, whether f_i and f_J are coprime and whether the former divides the latter.

We claim that

Lemma VII.7. For any monomial ideal, there is a monomial ideal sharing its cohomological support variety which can by taking a collection of subsets of $[n]$, assigning each a variable, and letting f_i be the product of the variables corresponding to the subsets containing i. Furthermore, for any minimal generating set of an equigenerated monomial ideal, there is a minimal generating set of the same size of an equigenerated monomial ideal sharing its cohomological support variety which can be constructed by taking a monomial ideal as described above, assigning to each variable a positive integer, and taking each variable to the power of its corresponding integer.

Proof. Consider some monomial ideal and some variable x_0 of its underlying ring. Let p_i be the power of x_0 in f_i. Then, recalling Lemma VII.6, we can replace $x_0^{p_i}$ with $\prod_{j=1}^{p_i} x_{0,j}$ for all i without modifying our cohomological support variety, and we can do so without affecting minimality as this is given in the monomial setting by the non-existence of one generator dividing another. After doing this for each variable in our underlying ring, we may either remove redundant variables, those which are contained in the same subset of $\boldsymbol{f}$ as others, proving our first claim, or replace them with the variables they copy, proving the second. □

As an example, after our first step the sequence $\boldsymbol{f} = (x_1x_2^2, x_2^3, x_3^3)$ would become

$$(x_{1,1}x_{2,1}x_{2,2}, x_{2,1}x_{2,2}x_{2,3}, x_{3,1}x_{3,2}x_{3,3}),$$

at which point our second steps would modify it into either

$$(x_{1,1}x_{2,1}, x_{2,1}x_{2,3}, x_{3,1}) \quad \text{or} \quad (x_{1,1}x_{2,1}^2, x_{2,1}^2x_{2,3}, x_{3,1}^3).$$

We categorize ideals by their corresponding greatest common denominator (GCD) graphs [BGP25, Definition 6.1]. As noted in [BGP25], there are a number of GCD graphs such that all corresponding monomial ideals must have full support:

Lemma VII.8 ([BGP25, Lemma 6.12])**.** Any monomial ideal whose GCD graph contains a vertex connected to every other vertex, or contains an edge such that every vertex is connected to exactly one of its vertices, has full support.

We use `myGraphs` to refer to the collection of graphs which is not guaranteed full support by this lemma.

In addition to these "forbidden graphs," we can also determine within these graphs "forbidden edges," where, if a monomial ideal constructed by Lemma VII.6 has this GCD graph, and if there is a variable corresponding to the two vertices in that edge, then our support is full. We define these forbidden edges in terms of "dense edges," such that each vertex in our graph is connected to at least one of its vertices:

Lemma VII.9. For any dense edge in our GCD graph of a monomial ideal without full support, there must be a third vertex whose monomial divides the LCM of the two comprising the dense edge in question.

Proof. If not, then the vertex of the Taylor graph corresponding to this edge is isolated, which guarantees full support by [BGP25, Lemma 6.9]. □

Corollary VII.10. For any dense edge in our GCD graph of a monomial ideal without full support, the set of variables corresponding to subsets disjoint from our dense edge must not cover all vertices connected to both vertices in our dense edge.

Proof. One of these vertices must divide the LCM of the vertices of the dense edge. □

Lemma VII.11. When $n = 6$ and $\boldsymbol{f}$ is minimal and consists of monomials, if we have two dense edges sharing a vertex, such that the two other vertices comprising these edges are not connected in our GCD graph, then we can have no variables corresponding to edges disjoint from both of these dense edges.

Proof. Say without loss of generality that we have dense edges connecting 1 to 2 and 3, and for contradiction that our monomial ideal has an variable corresponding to the edge connecting 4 and 5. Then by Corollary VII.10, our sixth monomial must be a factor of

$$\gcd\{\operatorname{lcm}\{f_{\underline{1}}, f_{\underline{2}}\}, \operatorname{lcm}\{f_{\underline{1}}, f_{\underline{3}}\}\}.$$

However, if $f_{\underline{2}}$ and $f_{\underline{3}}$ are coprime, this is simply $f_{\underline{1}}$, so $f_{\underline{6}}$ divides $f_{\underline{1}}$, contradicting minimality. □

We let `myCliques` refer to the set of cliques whose corresponding variables are not forbidden by Corollary VII.10 or Lemma VII.11.

We may now describe how our test file verifies Computation VII.5. Let `myMatrix`, over $\mathbb{Q}$, take each clique in `myCliques` to its corresponding zero-one n-vector. The preimage of the 1-space $\langle \sum_{i=1}^{n} \mathbf{e}_i \rangle$ contains all formal linear sums of cliques corresponding to equigenerated monomial ideals. Let `R` (whence our useful *rays* originate) be the intersection of this preimage with the space of formal linear sums with non-negative coefficients, and compute the primitive extreme rays of the resulting cone (those consisting of non-negative integers with no common factors; Macaulay2 returns these rays by default). Iteratively collect into `RAll` all sums of subsets of these rays which are not non-zero on a collection of cliques such that containing variables for each would imply full support by Corollary VII.10 or Lemma VII.11. By Lemma VII.7 and the fact that the cohomological support variety of a monomial ideal given by such a vector is determined by which entries are zero and which are non-zero, every non-full support of a cohomological support variety of an ideal with 6 equidegree monomial generators must be given by one of these vectors (up to order of course, due to our choices of permutations of graphs). Use `equigeneratedMonomialCSV` to compute the cohomological support varieties of the ideals corresponding to these vectors whenever the generating sets provided are minimal. Note that, thanks to our special attention to minimality in the above lemmas, this restriction to minimal generating sets does not remove any cohomological support varieties from our result which we set out to include.

4. Future work

This chapter and the last raise two natural questions.

Question VII.12. Can this strategy be leveraged to classify all monomial ideals with $n = 6$ generators, either manually or with computer assistance? What about for other fixed $n > 6$?

Question VII.13. Do all edge ideals of cycles in $4m + 2$ variables have cohomological support variety

$$\mathscr{V}(a_1 a_3 \cdots a_{4n+1} + a_2 a_4 \cdots a_{4m+2})?$$

The first question is much more difficult. That said, our treatment here does not address the strategies used in [BGP25] to help our classification by Taylor graphs in more than identifying those with full support, and it would be interesting to see whether using these two strategies in tandem is sufficient to answer it. Furthermore, a Macaulay2 method to perform our strategy for equigenerated monomial ideals has been implemented, but one to do so for non-equigenerated monomial ideals satisfying the conditions of Theorem VI.30 has not. Expansion of and experimentation with this method would be very useful in directing further inquiry. However, the second question, beyond what is known from Section 1, progress is scant.

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CITED LITERATURE

[AB00a] Avramov, L. & Buchweitz, R. Homological Algebra Modulo a Regular Sequence with Special Attention to Codimension Two. *Journal Of Algebra*. **230**, 24-67 (2000,8)

[AB00b] Avramov, L. & Buchweitz, R. Support varieties and cohomology over complete intersections. *Inventiones Mathematicae*. **142**, 285-318 (2000,11)

[ADE+] Abo, H., Decker, W., Eisenbud, D., Schreyer, F., Smith, G. & Stillman, M. BGG: Bernstein-Gelfand-Gelfand correspondence. Version 1.4.2. (A *Macaulay2* package available at `https://github.com/Macaulay2/M2/tree/master/M2/Macaulay2/packages`)

[AG02] Avramov, L. & Grayson, D. Resolutions and Cohomology over Complete Intersections. *Computations In Algebraic Geometry With Macaulay 2*. pp. 131-178 (2002)

[AKM88] Avramov, L., Kustin, A. & Miller, M. Poincaré series of modules over local rings of small embedding codepth or small linking number. *Journal Of Algebra*. **118**, 162-204 (1988,10)

[Ale17] Alesandroni, G. Minimal resolutions of dominant and semidominant ideals. *Journal Of Pure And Applied Algebra*. **221**, 780-798 (2017,4)

[Avr13] Avramov, L. (Contravariant) Koszul Duality for DG Algebras. *Algebras, Quivers And Representations*. pp. 13-58 (2013)

[Avr86] Avramov, L. Golod homomorphisms. *Algebra, Algebraic Topology And Their Interactions*. pp. 59-78 (1986)

[Avr89] Avramov, L. Modules of finite virtual projective dimension. *Inventiones Mathematicae*. **96**, 71-101 (1989,2)

[Avr98] Avramov, L. Infinite Free Resolutions. *Six Lectures On Commutative Algebra*. pp. 1-118 (1998)

[BBG+] Banks, M., Brown, M., Gomes, T., Sridhar, P., Davila, E. & Zotine, S. MultigradedBGG: the multigraded BGG correspondence and differential modules. Version 1.2. (A *Macaulay2* package available at https://github.com/Macaulay2/M2/tree/master/M2/Macaulay2/packages)

[BBL+13] Bhatt, B., Blickle, M., Lyubeznik, G., Singh, A. & Zhang, W. Local cohomology modules of a smooth $\mathbb{Z}$-algebra have finitely many associated primes. *Inventiones Mathematicae*. **197**, 509-519 (2013,11)

[BCL+25] Briggs, B., Cameron, J., Letz, J. & Pollitz, J. Koszul homomorphisms and universal resolutions in local algebra. *Forum Of Mathematics, Sigma*. **13** (2025)

[Ber07] Bergh, P. On Support Varieties for Modules over Complete Intersections. *Proceedings Of The American Mathematical Society*. **135**, 3795-3803 (2007)

[BGG78] Bernshtein, I., Gel'fand, I. & Gel'fand, S. Algebraic bundles over P^n and problems of linear algebra. *Functional Analysis And Its Applications*. **12** pp. 212-214 (1978)

[BGP21] Briggs, B., Grifo, E. & Pollitz, J. Constructing nonproxy small test modules for the complete intersection property. *Nagoya Mathematical Journal*. **246** pp. 412-429 (2021,6)

[BGP24] Briggs, B., Grifo, E. & Pollitz, J. Bounds on cohomological support varieties. *Transactions Of The American Mathematical Society, Series B*. **11**, 703-726 (2024,3)

[BGP25] Briggs, B., Grifo, E. & Pollitz, J. The embedded deformation problem for monomial ideals. (arXiv,2025)

[BMFM19] Buijs, U., Moreno-Fernández, J. & Murillo, A. A_∞ Structures and Massey Products. *Mediterranean Journal Of Mathematics*. **17** (2019,12)

[BRS00] Brodmann, M., Rotthaus, C. & Sharp, R. On annihilators and associated primes of local cohomology modules. *Journal Of Pure And Applied Algebra*. **153**, 197-227 (2000,11)

[Bur15] Burke, J. Higher homotopies and Golod rings. (arXiv,2015)

[Bur18] Burke, J. Transfer of A-infinity structures to projective resolutions. (arXiv,2018)

[CFH24] Christensen, L., Foxby, H. & Holm, H. Derived Category Methods in Commutative Algebra. *Springer Monographs In Mathematics*. (2024)

[DE22] Dao, H. & Eisenbud, D. Linearity of free resolutions of monomial ideals. *Research In The Mathematical Sciences*. **9** (2022,5)

[EFS03] Eisenbud, D., Fløystad, G. & Schreyer, F. Sheaf cohomology and free resolutions over exterior algebras. *Transactions Of The American Mathematical Society*.

355, 4397-4426 (2003,7)

[Eis80] Eisenbud, D. Homological algebra on a complete intersection, with an application to group representations. *Transactions Of The American Mathematical Society*. **260**, 35-64 (1980)

[ES] Eisenbud, D. & Stillman, M. AInfinity: AInfinity structures on free resolutions. Version 0.1. (A *Macaulay2* package available at `https://github.com/Macaulay2/M2/tree/master/M2/Macaulay2/packages`)

[Frö75] Fröberg, R. Determination of a class of Poincaré series. *Mathematica Scandinavica*. **37**, 29-39 (1975)

[GLP] Grifo, E., Letz, J. & Pollitz, J. ThickSubcategories: Computing levels of complexes and support varieties of complexes. Version 1.2. (A *Macaulay2* package available at `https://github.com/eloisagrifo/levels/tree/master`)

[Gro68] Grothendieck, A. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux, 1961-1962 (Local cohomology of coherent sheaves and global and local Lefschetz theorems), North Holland 1968

[GS] Grayson, D. & Stillman, M. Macaulay2, a software system for research in algebraic geometry. (Available at `http://www2.macaulay2.com`)

[Gul74] Gulliksen, T. A change of ring theorem with applications to Poincaré series and intersection multiplicity. *Mathematica Scandinavica*. **34**, 167-183 (1974)

[GZ25] Gintz, M. & Zhang, W. Koszul cohomology and support of local cohomology modules of complete intersections. (arXiv,2025), to appear in *Journal of the London Mathematical Society*

[Hel01] Hellus, M. On the Set of Associated Primes of a Local Cohomology Module. *Journal Of Algebra*. **237**, 406-419 (2001,3)

[HKM09] Huneke, C., Katz, D. & Marley, T. On the support of local cohomology. *Journal Of Algebra*. **322**, 3194-3211 (2009,11)

[HN17] Hochster, M. & Núñez-Betancourt, L. Support of local cohomology modules over hypersurfaces and rings with FFRT. *Mathematical Research Letters*. **24**, 401 - 420 (2017)

[HS93] Huneke, C. & Sharp, R. Bass numbers of local cohomology modules. *Transactions Of The American Mathematical Society*. **339**, 765-779 (1993)

[Hun92] Huneke, C. Problems on local cohomology. *Free resolutions in commutative algebra and algebraic geometry*. (Sundance, Utah, 1990), pp. 93-108 (1992)

[Kad05] Kadeishvili, T. On the homology theory of fibre spaces. (2005)

[Kad82] Kadeishvili, T. Algebraic structure in the homology of an A_∞-algebra. *Soobshch. Akad. Nauk. Gruz. SSR.* **108** pp. 249-252 (1982)

[Kat02] Katzman, M. An example of an infinite set of associated primes of a local cohomology module. *Journal Of Algebra.* **252**, 161-166 (2002,6)

[Kel01] Keller, B. Introduction to A-infinity algebras and modules. *Homology, Homotopy And Applications.* **3**, 1 - 35 (2001)

[KS99] Khashyarmanesh, K. & Salarian, S. On the associated primes of local cohomology modules. *Communications In Algebra.* **27**, 6191-6198 (1999,1)

[Kun69] Kunz, E. Characterizations of Regular Local Rings of Characteristic p. *American Journal Of Mathematics.* **91**, 772 (1969,7)

[KZ17] Katzman, M. & Zhang, W. The Support of Local Cohomology Modules. *International Mathematics Research Notices.* (2017,5)

[Koz99] Kozlov, D. Complexes of Directed Trees. *Journal Of Combinatorial Theory, Series A.* **88**, 112-122 (1999,10)

[Lev75] Levin, G. Local rings and Golod homomorphisms. *Journal Of Algebra.* **37**, 266-289 (1975,11)

[LL21] Löfwall, C. & Lundqvist, S. Software for doing computations in graded Lie algebras. *Journal Of Software For Algebra And Geometry.* **11**, 9-14 (2021,12)

[LPW09] Lu, D., Palmieri, J., Wu, Q. & Zhang, J. A-infinity structure on Ext-algebras. *Journal Of Pure And Applied Algebra.* **213**, 2017-2037 (2009,11)

[Lyu00] Lyubeznik, G. Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case. *Communications In Algebra.* **28**, 5867-5882 (2000,1)

[Lyu93] Lyubeznik, G. Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra). *Inventiones Mathematicae.* **113**, 41-55 (1993,12)

[Lyu97] Lyubeznik, G. F-modules: applications to local cohomology and D-modules in characteristic $p > 0$. *Journal Für Die Reine Und Angewandte Mathematik (Crelles Journal).* **1997**, 65-130 (1997,10)

[Moo] Moore, F. DGAlgebras: Data type for DG algebras. Version 1.1.1–with fix of killCycles and new code and docs for displayBlockDiff and blockDiff. (A

Macaulay2 package available at `https://github.com/Macaulay2/M2/tree/master/M2/Macaulay2/packages`)

[Pol19] Pollitz, J. The derived category of a locally complete intersection ring. *Advances In Mathematics*. **354** pp. 106752 (2019,10)

[Pol21] Pollitz, J. Cohomological supports over derived complete intersections and local rings. *Mathematische Zeitschrift*. **299**, 2063-2101 (2021,5)

[Pri70] Priddy, S. Koszul resolutions. *Transactions Of The American Mathematical Society*. **152**, 39-60 (1970)

[Pro11] Prouté, A. A_∞-structures. Modèles minimaux de Baues-Lemaire et Kadeishvili et homologie des fibrations. *Repr. Theory Appl. Categ.* **21** pp. 1-99 (2011)

[Sch18] Schnibben, T. (2018). Local Rings and Golod Homomorphisms. (Doctoral dissertation).

[Sha69] Shamash, J. The Poincaréseries of a local ring. *Journal Of Algebra*. **12**, 453-470 (1969,8)

[Sin00] Singh, A. p-torsion elements in local cohomology modules. *Mathematical Research Letters*. **7**, 165 - 176 (2000)

[SS04] Singh, A. & Swanson, I. Associated primes of local cohomology modules and of Frobenius powers. *International Mathematics Research Notices*. **2004**, 1703 (2004)

[Sta63] Stasheff, J. Homotopy Associativity of H-Spaces. I. *Transactions Of The American Mathematical Society*. **108**, 275 (1963,8)

[Ste13] Stevenson, G. Subcategories of singularity categories via tensor actions. *Compositio Mathematica*. **150**, 229-272 (2013,11)

[Tay66] Taylor, D. Ideals Generated By Monomials In An R-sequence (Order No. T-13006). (1966), https://www.proquest.com/dissertations-theses/ideals-generated-monomials-r-sequence/docview/302227382/se-2, Available from Dissertations & Theses Big Ten Academic Alliance; ProQuest Dissertations & Theses Global; ProQuest Dissertations & Theses Global Closed Collection. (302227382).

[Val12] Vallette, B. Algebra+Homotopy=Operad. (arXiv,2012)

[Wei94] Weibel, C. An Introduction to Homological Algebra. (Cambridge University Press,1994,4)

[Yuz99] Yuzvinsky, S. Taylor and minimal resolutions of homogeneous polynomial ideals. *Mathematical Research Letters*. **6**, 779-793 (1999), http://dx.doi.org/10.4310/MRL.1999.v6.n6.a14

[Zha15] Zhang, Y. Four Variations on Graded Posets. *Discrete Mathematics & Theoretical Computer Science*. **DMTCS Proceedings, 27th...** (2015,1), http://dx.doi.org/10.46298/dmtcs.2492

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	Gintz, M., Keller, T., Kortje, M., Wang, Z. & Yang, Y. The number of set orbits of permutation groups and the group order. *Bulletin Of The Australian Mathematical Society.* **106**, 89-101 (2022,1)