Square Factors in Projective Character Degrees

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Introduction

For a finite group G, let $\mathrm{cd}(G)$ be the multi-set of irreducible character degrees of G. In the past few decades, people have discovered that some information about G can be extracted from sufficiently nice sets $\mathrm{cd}(G)$:

- If G is non-solvable and $\mathrm{cd}(G)$ consists of prime powers, then for some abelian A we have $G \cong A_5 \times A$ or $G \cong \mathrm{PSL}_2(8) \times A$ [11],
- If G is non-solvable and $\mathrm{cd}(G)$ consists of square-free integers, then for some solvable R we have $G \cong A_7 \times R$ [4].

These results revolve around the ordinary representations of G, often simply called the representations of G. We wish to strengthen these results with more general structures, known as projective representations.

What Are Projective Character Degrees?

A *projective representation* of a group G on a vector space V over a field F is a map $\rho : G \to \mathrm{GL}(V)$ with a function $\alpha : G^2 \to F^\times$ such that for all $a, b \in G$ we have

$$\rho(a)\rho(b) = \rho(ab)\alpha(a, b).$$

The function α is known as the *factor set* of ρ. Two factor sets are *equivalent* if there is some $\mu : G \to F^\times$ such that

$$\beta(a, b) = \alpha(a, b)\mu(a)\mu(b)\mu(ab)^{-1}.$$

Two projective representations are *equivalent* if we can get to one from the other by adjusting the equivalent factor sets and conjugating by some matrix. An *irreducible* projective representation is not equivalent to one expressible as $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$.The set of degrees of irreducible projective representations of G with factor set α is written as $\mathrm{cd}_\alpha(G)$.

Thus if we consider $\mathrm{cd}_\alpha(G)$ for arbitrary α rather than only $\mathrm{cd}_1(G) = \mathrm{cd}(G)$, we can immediately propose a generalization to the theorems described above. The first of these has already been proved in [6], and we wish to use the logic of that argument to prove the other:

Conjecture 1. If G is a non-solvable finite group and α is a factor set of G such that all values in $\mathrm{cd}_\alpha(G)$ consists of square-free integers, then for some solvable R we have $G \cong A_7 \times R$.

Since ordinary characters have been studied more thoroughly and for a longer period of time, it will be useful in many cases to consider projective representations through the lens of ordinary representations. Therefore, the following theorem will prove invaluable:

Theorem 1 ([5])**.** *For a finite group G, there is a finite group Γ with center L satisfying the following properties:*

- *The group Γ/L is congruent to G,*
- *There is a bijection from the factor sets α of G to the irreducible representations λ of L satisfying*

$$cd_\alpha(G) = \{\gamma(1)/\lambda(1) \mid \gamma \in Irr(\Gamma \mid \lambda)\},$$

where

$$Irr(\Gamma \mid \lambda) = \{\gamma \in Irr(\Gamma) \mid [\gamma_L, \lambda] \neq 0\}.$$

This group is known as the Schur cover of G.

Proposed Method of Proof

Using the Schur cover, we can instead prove the following conjecture:

Conjecture 2. Let L be a normal central subgroup of the finite Γ, and let λ be an irreducible representation of L. Then if all $\gamma(1)/\lambda(1)$ for $\gamma \in \mathrm{Irr}(\Gamma \mid \lambda)$ are square-free, then Γ/L is solvable or expressible as $R \times A_7$ for R solvable.

The following lemma is instrumental to our proposed proof:

Lemma 1 ([5])**.** Let Γ be a finite group, let λ be an irreducible character of $L \trianglelefteq \Gamma$, and let $I_\Gamma(\lambda)$, known as the *inertia subgroup* of λ, be the subgroup of Γ fixing λ. Then the set of $\gamma(1)/\lambda(1)$ such that $[\gamma_L, \lambda] \neq 0$ is identical to $|\Gamma : I_\Gamma(\lambda)|\mathrm{cd}_\alpha(I_\Gamma(\lambda)/L)$ for some α.

In following with [6], our general strategy is as follows:

- assume that we have some contradiction which minimizes $|\Gamma/L|$, and let N be the largest normal solvable subgroup of Γ. Show first that Γ/N is simple,
- then show that some $\nu \in \mathrm{Irr}(N \mid \lambda)$ is Γ-invariant, so that by Lemma 1 we can have Γ/L simple,
- and finally use the classification of finite simple groups to consider the remaining possibilities.

Proof of Simplicity of Γ/N

Say Γ/N is not simple, and let M/N be a minimal normal subgroup of Γ/N. By Lemma 1, the minimality of $|\Gamma/L|$, and the induction of the relevant representations, we have $M/L \cong N/L \times A_M/L$ for $A_M/L \cong A_7$. If M is not unique, then there is another minimal K/N, so that Γ/L has a normal subgroup isomorphic to $A_7 \times A_7$. By [9], given some $H \trianglelefteq G$ and some $\eta \in \mathrm{Irr}(H)$, the values of $\mathrm{cd}(G \mid \eta)$ are all coprime with p for some prime p only if the Sylow p-subgroup of G/H is abelian, contradiction, so M is unique and by the Normalizer-Centralizer theorem we have $G/N \cong S_7$. By [3] there is some $\mu \in \mathrm{Irr}(A_M \mid \lambda)$ with $\mu(1) = 10\lambda(1)$. Applying [9] once more and using Lemma 1, there is an automorphism of A_M allowing us to construct $A_M \rtimes Z_2$ where $(A_M \rtimes Z_2)/L \cong S_7$. Since cyclic groups only have trivial factor sets there is a projective representation of S_7 of degree 10 contradicting [3].

Progress for Simplicity of Γ/L

Recall we need to show that some $\nu \in \mathrm{Irr}(N \mid \lambda)$ is Γ-invariant, which is to say $I_\Gamma(\nu) = \Gamma$. Our plan for showing that Γ/L is simple is roughly as follows:

- use the information from [9] to narrow the instances where $|\Gamma : I_\Gamma(\nu)|$ is even,
- use classifications of odd-index subgroups of finite simple groups for the remaining cases.

Progress has been promising in both cases given the wealth of existing knowledge on the subgroups in question, though we expect to be able to use the same information from [9] to eliminate characteristics of Lie-type groups from $|G : I_\Gamma(\nu)|$ as well.

Families of Finite Simple Groups Γ/L Cannot Be Isomorphic To

For $n > 7$, there is only one non-trivial factor set, and [10] describes a projective representation of A_n with this factor set and degree a multiple of 4 for $n > 7$. The cases $n < 7$ can be checked from [3] by hand, though A_7 has not yet been considered. The projective character degree sets of the sporadic groups can be shown to have square factors with [3]. The groups of Lie types E_6 and E_7 are arguably the most interesting cases which have been shown to have square factors. The argument relies on the following lemma, which is a generalization of a well-known fact about irreducible ordinary representations:

Lemma 2 ([2])**.** The sum of the squares of the elements of $\mathrm{cd}_\alpha(G)$ is $|G|$.

This leads to a method of proof for showing that a group's projective character degree sets all have elements with square factors. For groups of Lie types E_6 and E_7 over finite fields F_q, the character degree sets of their Schur covers are available in [7], making this a complete argument.

Finding Square Factors with a Weighted Pigeonhole Argument

Say that a group G with Schur cover Γ has n factor sets up to equivalence. Let $\mathrm{cd}_{sq}(\Gamma)$ be the set of elements of $\mathrm{cd}(\Gamma)$ which have square factors. Then by Lemma 2, in order to show that all projective character degree sets of G have elements with square factors, it suffices to show that the sum of the squares of the elements of $\mathrm{cd}_{sq}(\Gamma)$ is greater than $\frac{(n-1)|\Gamma|}{n}$.

Future Work: Other Lie-Type Groups

For the remaining finite simple groups, which are not yet confirmed to obey the conjecture, we have two potential strategies in mind. First, the Schur covers of most finite simple Lie-type groups are well-studied Lie groups over finite fields.

Applying Deligne-Lusztig Theory

Deligne-Lusztig theory partitions the irreducible representations of these Schur covers by the conjugacy classes of elements known as *semisimple* within a related group known as the *dual* of the original. Importantly, the degrees of the corresponding elements are multiples of the indices of the centralizers of these semisimple elements [1]. That is, we need to show two things:

- most of these indices have square factors,
- the square sums of the degrees in the partitions are roughly equal.

Already known for some of these groups are the formulaue for these square sums [8] and the structure of these centralizers [1], and we need only to synthesize this information.

The groups for which this information is missing or very dense have at most 6 factor sets, allowing us to find a projective representation with square-factor degree for each factor set. There are many known classes of representations of these Schur covers, such as the Weyl representations, which may potentially lead to such projective representations.

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