Two Approaches to the Class Field Tower Problem

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Abstract

We consider two sets of topics relevant to discussion surrounding the class field tower problem, one historical and one modern. The first chapter gives an introduction to profinite groups and group cohomology, culminating in the presentation of an existing proof of the Golod-Shafarevich theorem, which was the final result required to show the existence of infinite class field towers. The second chapter considers a proposal to construct class field towers from sequences of diffeomorphism groups of manifolds, via C^*-algebras and dimension groups. We survey the relevant topics, inspect the existing proposal and suggest procedures which may improve it.

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I couldn't have done this by myself.

Declaration

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Contents

Chapter 1

The Golod-Shafarevich Theorem

Given a number field K, its *Hilbert class field* $\mathscr{H}(K)$ is the maximal abelian unramified extension of K. It has the properties that every ideal of $\mathcal{O}_K$ extends to a principal ideal of $\mathcal{O}_{\mathscr{H}(K)}$, and $\mathrm{Gal}(\mathscr{H}(K)/K)$ is isomorphic to the class group of K. The *class field tower problem* is closely related to this structure. It considers for an arbitrary number field K the sequence of number fields

$$K = \mathscr{H}^0(K) \subseteq \mathscr{H}^1(K) \subseteq \mathscr{H}^2(K) \subseteq \cdots ,$$

known as the *class field tower* of K, where $\mathscr{H}^i$ is the composition of i copies of $\mathscr{H}$. In particular, the question asks if this sequence stabilizes: that is, if there is some non-negative integer i such that $\mathscr{H}^i(K) = \mathscr{H}^{i+1}(K)$. The Golod-Shafarevich theorem states that this claim is not always true, and provides number fields whose class field towers do not stabilize. This theorem was proven using Galois cohomology and the profinite group structure associated with the Hilbert class field tower. We provide a proof of the theorem to familiarize the reader with some of the existing scholarship surrounding the problem. We note however that while the result of the Golod-Shafarevich theorem concerns class field towers, it does so by citing a result stating a cohomological fact regarding their profinite group structure, which we cite without proof, so this chapter will contain much less context surrounding the class field towers themselves than the title of this thesis might imply. Section 1.1 pulls largely from [DDSMS99], whereas Sections 1.2, 1.3, and 1.4 are adapted primarily from [NSW08], though they are supplemented by [Sha], [Zho17], and [Ser97]. The result pulled from [Š63], which is in Russian, can be found in English in [Zho17].

1.1 Profinite groups

In order for us to extract useful information from the cohomology structures of the Galois groups $\mathrm{Gal}(\mathscr{H}^{i+1}(K)/\mathscr{H}(K))$, we will need to utilize their profinite group structure. In order to do this, we describe relevant information regarding profinite groups here.

Definition 1.1. Given a group G, we can define a map from G to G by inversion, and a map from $G \times G$ to G by multiplication. A *topological group* is a group with a topological structure on its elements such that these maps are both continuous.

Definition 1.2. Consider a set $\{G_i \mid i \in I\}$ of finite groups for some ordered set I with ordering $\leq$ such that for all $i, j \in I$ there is some $k \geq i, j$. Say further that we have a homomorphism $\phi_{i,j} : G_i \to G_j$ for all $i \geq j$. The *inverse limit* of this system of groups is the topological group satisfying the following conditions:

- The group consists of the elements of $\prod_{i \in I} G_i$ such that if $i \geq j$ then the image of the G_i-coordinate under $\phi_{i,j}$ is the G_j-coordinate.
- The topological structure can be constructed by giving each G_i the discrete topology, imposing the product topology on $\prod_{i \in I} G_i$, and taking the relative topology to yield a topological structure on our group.

A *profinite group* is a topological group expressible as an inverse limit of a system of finite groups. A *pro-p group* is a topological group expressible as an inverse limit of a system of p-groups.

Theorem 1.3. *A profinite group G is compact, Hausdorff, and totally disconnected.*

Proof. Let $\{G_i \mid i \in I\}$ be our relevant set of finite groups. By Tychonoff's theorem, which states that the product of compact spaces under the product topology is compact, we have that $G' := \prod_{i \in I} G_i$ is compact. To show that it is Hausdorff and totally disconnected, note that the set of elements with a given value of a given coordinate is open, and for distinct elements a, b we can find a coordinate corresponding to a group $G_{i_{(a,b)}}$ at which their values differ. We can then partition $G_{i_{(a,b)}}$ into two sets so that the elements corresponding to a and b are in different sets, and then partition G' into two disjoint sets based on the values of the coordinates corresponding to $G_{i_{(a,b)}}$. These sets will be open as they are unions of sets corresponding to individual elements of $G_{(a,b)}$.

Now that we have shown that G' is totally disconnected, Hausdorff, and compact, we wish to do the same for G. Since G takes the relative topology of $\prod_{i \in I} G_i$, it is also totally disconnected and Hausdorff. To show that it is compact, we must show that it is a closed subset of G'. Consider some $g \in G' \setminus G$. Since $g \notin G$, there are some i, j with $i \geq j$ such that $\phi_{i,j}$ of the G_i-coordinate of g does not yield the G_j-coordinate. The set of elements of G' sharing G_i- and G_j-coordinates with g is open and disjoint from G, proving that G is closed, and is thus compact. □

Remark. The converse of this theorem is also true: that is, all compact, Hausdorff, and totally disconnected groups are profinite. Some texts will define a profinite group as one which satisfies these three topological properties, so it is useful to be aware that these definitions are equivalent.

Definition 1.4. A *topological generating set* of a topological group G is a subset S of G such that every neighborhood of the identity contains all but at most finitely many elements of S and the subgroup generated by S is dense in G. A topological group is *topologically finitely generated* if it has a finite topological generating set.

Definition 1.5. A set S generates a normal subgroup N of a topological group G *as a normal subgroup* if the elements in conjugacy classes of G which non-trivially intersect S form a topological generating set of N when it is given the induced topology.

Definition 1.6. The *Frattini subgroup* $\Phi(G)$ of a pro-p group G is the maximal closed subgroup of G.

Lemma 1.7. *For a pro-p group G, the subgroup $\Phi(G)$ is the closure of the subgroup generated by the p-th powers and commutators of G.*

Theorem 1.8 (Burnside's basis theorem)**.** *A subset S of a pro-p group G such that every neighborhood of the identity contains all but at most finitely many elements of S is a topological generating set of G if and only if it maps to a topological generating set of the group $G/\Phi(G)$ with the quotient topology.*

To clarify Burnside's basis theorem, and to make more effective use of the Frattini subgroup, we introduce the following

Theorem 1.9. *If a pro-p group G is topologically finitely generated, then every subgroup of G of finite index is open.*

Remark. Since the proof of the Golod-Shafarevich theorem that we are building towards here, it has been shown in [NS07] that all subgroups of finitely generated profinite groups of finite index are open as well.

Corollary 1.10. *Given a topologically finitely generated pro-p group G, the quotient topology on $G/\Phi(G)$ is discrete.*

Definition 1.11. The *profinite* (resp., *pro-p*) *completion* of a discrete group G is the inverse limit of quotients of G over its open normal subgroups of finite (resp., p-power) index, where p is a prime where defined.

Definition 1.12. The *free profinite* (resp., *pro-p*) *group* on a set S is the inverse limit of the quotients of the free group with generating set S over its normal subgroups with finite (resp., p-power) index which contain all but at most finitely many elements of S, where p is a prime where defined.

Theorem 1.13. *Any profinite (or pro-p) group G which is topologically generated by a set S is a quotient of the free profinite (resp., pro-p) group on S.*

1.2 Group cohomology

The proof of the Golod-Shafarevich theorem relies heavily on the cohomological properties of the groups $\mathrm{Gal}(\mathscr{H}^{i+1}(K)/\mathscr{H}^{i}(K))$, and as such we begin by introducing some relevant introductory terminology from group cohomology. In this section, we let G be a topological group and A be a topological group which is also a G-module.

Definition 1.14. Let n be a non-negative integer.

(i) The set $X^n = X^n(G, A)$ consists of all continuous group homomorphisms from G^{n+1} to A, regardless of whether they respect the group action. We impose a group operation on X^n given by addition. Furthermore, we impose an action of G on X^n such that if $\sigma, \sigma_0, \dots, \sigma_n \in G$ and $x \in X^n$ we have

$$(\sigma x)(\sigma_0, \dots, \sigma_n) = \sigma x(\sigma^{-1}\sigma_0, \dots, \sigma^{-1}\sigma_n).$$

(ii) The *n-th differential* $\partial = \partial^n = \partial^n_A : X^{n-1} \to X^n$ is a G-homomorphism given for $i \geq 1$, $x \in X^n$, and $\sigma_0, \dots, \sigma_n \in G$ by

$$(\partial x)(\sigma_0, \dots, \sigma_n) = \sum_{i=0}^{n} (-1)^i x(\sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_n),$$

where $\hat{\sigma}_i$ indicates that σ_i has been omitted from the relevant $(n+1)$-tuple. We also define the 0-th differential $\partial^0_A : A \to X^0$, which takes an item $a \in A$ to the constant function taking all of G to a.

Theorem 1.15. *The sequence*

$$0 \to A \to X^0 \to X^1 \to X^2 \to \cdots,$$

known as the standard resolution *of A, is exact.*

Proof. We begin by showing that we always have $\partial\partial = 0$. By an explicit calculation, we have $\partial^1\partial^0 = 0$. For $i > 0$, we have $\partial^i\partial^{i-1}(x)$ equal to a sum of terms of the form $x(\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \hat{\sigma}_j, \ldots, \sigma_i)$. In particular, each such summand is seen twice, once with coefficient $(-1)^i(-1)^j$ and once with coefficient $(-1)^i(-1)^{j-1}$, giving a sum of zero.

For exactness, we consider the map $D^{-1} : X^0 \to A$ taking a map x to $x(1)$, along with the maps $D^n : X^{n+1} \to X^n$ for $i \geq 0$, which for $\sigma_0, \ldots, \sigma_n \in G$ and $x \in X^{n+1}$ satisfy

$$(D^n x)(\sigma_0, \ldots, \sigma_n) = x(1, \sigma_0, \ldots, \sigma_n).$$

By an explicit calculation, for $n \geq 0$ we have that $D^n \circ \partial^{n+1} + \partial^n \circ D^{n-1}$ is the identity map. Therefore, if $x \in \ker \partial^{n+1}$, then $x = \partial^n \cdot D^{n-1}(x)$, which is to say $x \in \operatorname{im} \partial^n$. This, along with the fact that $\partial\partial = 0$, is sufficient to prove the claim. $\square$

Definition 1.16. Let G be a group, A be a G-module, and n be a non-negative integer.

(i) The group $C^n = C^n(G, A) := X^n(G, A)^G$ is called the group of *homogeneous n-cochains* of G with coefficients in A.

(ii) The group $Z^n = Z^n(G, A) := \ker \partial^{n+1} \cap C^n$ is called the group of *n-cocycles* of G with coefficients in A.

(iii) The group $B^n = B^n(G, A)$, equal to 0 when $n = 0$ and $\partial^n(C^{n-1})$ otherwise, is called the group of *n-coboundaries* of G with coefficients in A.

(iv) The group $H^n(G, A) := Z^n(G, A)/B^n(G, A)$ is *n-th cohomology group* of G with coefficients in A.

Remark. Some texts will instead consider *inhomogeneous cochains*, which are isomorphic to their corresponding homogemeous cochains but are written slightly differently. These are often more useful for computation, as they consider fewer inputs than their homogeneous counterparts, but are much more cumbersome, and will not be particularly useful to us here.

Example 1.17. The group $H^0(G, A)$ is the group A^G of elements of A which are invariant under the action of G. That is, it is isomorphic to $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$, where G acts trivially on $\mathbb{Z}$.

Theorem 1.18. *If G acts trivially on A, then $H^1(G, M) \cong \mathrm{Hom}_{\mathrm{cts}}(G, A)$.*

Proof. Note first that Z^1 consists of continuous functions $x : G^2 \to A$ in C^1 such that for $\sigma_0, \sigma_1, \sigma_2 \in G$ we have $x(\sigma_0, \sigma_1) + x(\sigma_1, \sigma_2) = x(\sigma_0, \sigma_2)$. Since x is invariant under the action of G, this is to say that $x(1, \sigma_0^{-1}\sigma_1) + x(1, \sigma_1^{-1}\sigma_2) = x(1, \sigma_0^{-1}\sigma_2)$. From this it is clear that this is isomorphic to $\mathrm{Hom}_{\mathrm{cts}}(G, A)$. Furthermore, we see that B^1 consists of functions in C^1 from G^2 to A expressible as $x(\sigma_1) - x(\sigma_2)$ for some $x \in X^0$. However, every element of X^0 is G-invariant, implying that $x(\sigma) = x(1)$ for $\sigma \in G$. Thus B^1 is trivial, completing the proof. $\square$

As is often the case with this sort of cohomological structure, we have a long exact sequence containing the groups H^n for $n > 0$:

Theorem 1.19. *Given an exact sequence* $0 \to A \to B \to C \to 0$ *of* G*-modules, we have the long exact sequence*

$$\begin{aligned} &0 \to A^G \to B^G \to C^G \to H^1(G, A) \to \cdots \\ &\qquad \cdots \to H^n(G, A) \to H^n(G, B) \to H^n(G, C) \to H^{n+1}(G, A) \to \cdots . \end{aligned}$$

Definition 1.20. Let A be a G-module. We attach an addition operation and an action of G on the set of maps from G to A. We give addition by addition, and for $\sigma, \tau \in G$ and x a map from G to A, we let $(\sigma x)(\tau) = \sigma x(\sigma^{-1}\tau)$. The resulting G-module is written as $\mathrm{Ind}_G(A)$, and any G-module expressible in this way is known as an *induced* G*-module.*

An explicit calculation confirms that the maps from G-modules A to their induced G-modules $\mathrm{Ind}_G(A)$, given by taking $a \in A$ to a map taking the identity in G to a and the rest of G to zero, comprise a functor. This functor can also be shown to be exact.

Theorem 1.21. *For* A *a* G*-module and* $n > 0$*, the cohomology group* $H^n(G, \mathrm{Ind}_G(A))$ *is trivial.*

Proof. Consider the map $X^n(G, \mathrm{Ind}_G(A))^G \to X^n(G, A)$ for $n \geq 0$, given by the map $A \hookrightarrow \mathrm{Ind}_G(A)$, which takes $a \in A$ to a map taking the identity in G to a and the rest of G to zero. It is clear that this map commutes with the corresponding differential maps. Furthermore, it is an isomorphism, as we can give an inverse which takes a map $x \in X^n(G, A)$ to a map which takes $(\sigma_0, \ldots, \sigma_n) \in G^{n+1}$ to a map which takes $\sigma \in G$ to $\sigma x(\sigma^{-1}\sigma_0, \ldots, \sigma^{-1}\sigma_n)$. Therefore the standard resolution of A can be written as

$$0 \to A \to C^0(G, \mathrm{Ind}_G(A)) \to C^1(G, \mathrm{Ind}_G(A)) \to \cdots ,$$

with the boundary maps between C^{n-1} and C^n for $n > 0$ given by the corresponding differentials. However, by Theorem 1.15, this is exact, and the definition of $H^n(G, \mathrm{Ind}_G(A))$ gives us our result. $\square$

The following theorem will also prove useful to our understanding of the material:

Lemma 1.22. *If G is a finite p-group and A is a finite G-module such that $pA = 0$. Then $A^G = 0$ implies $A = 0$.*

Proof. Note that for $a \in A \setminus A^G$, the orbit of a under the action of G has order a multiple of p. Thus the order of A modulo p is equal to the order of A^G modulo p. If A^G is trivial, then it has order 1, so by the fact that A has order a power of p, it must be trivial as well. □

1.3 Generator and relation ranks

An important context for the use of group cohomology in profinite and other topological groups arises from two basic properties of such groups, namely *generator ranks* and *relation ranks.*

Definition 1.23. The *generator rank* $d(G)$ of a topological group G is the smallest order of a topological generating set of G.

From here, whenever we consider $\mathbb{F}_p$ as a G-module, we impose upon it a trivial G-module structure.

Theorem 1.24. *If G is a topologically finitely generated pro-p group, then $d(G) = \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$.*

Proof. By Theorem 1.18, we have $H^1(G, \mathbb{F}_p) \cong \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{F}_p)$. Furthermore, for $f \in \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{F}_p)$, we clearly have $f(g) = 0$ for $g \in \Phi(G)$, implying that $\mathrm{Hom}_{\mathrm{cts}}(G/\Phi(G), \mathbb{F}_p) \cong \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{F}_p)$. Note that $G/\Phi(G)$ is an elementary abelian p-group, that is, a vector space over $\mathbb{F}_p$, so by Corollary 1.10 we have that $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ is the dimension of the dual of $G/\Phi(G)$. By Theorem 1.8, Corollary 1.10, and our assumption that G is topologically finitely generated, the dimension of $G/\Phi(G)$ must be finite, so that the dimension of $G/\Phi(G)$ must be that of its dual, and the subgroup generated by a minimal topological generating set of $G/\Phi(G)$ must be $G/\Phi(G)$. Thus the minimal topological generating sets of $G/\Phi(G)$ are the minimal generating sets of $G/\Phi(G)$, and by Theorem 1.8 once more we have that $d(G)$ is the dimension of $G/\Phi(G)$, so that both of the quantities in question are equal to the dimension of $G/\Phi(G)$, proving the claim. □

By Theorem 1.13, for a pro-p group G there is some R for which there exists a short exact sequence

$$0 \to R \to F \to G \to 0,$$

where F is a free pro-p group on a minimal topological generating set of G.

Definition 1.25. A short exact sequence

$$0 \to R \to F \to G \to 0$$

such that F is a free pro-p group on a minimal topological generating set of G, which must exist by Theorem 1.13, is known as a *presentation* of G by F. A set which generates R as a normal subgroup of F is known as a *system of relations* of G.

Before continuing, we state without proof three relevant lemmas.

Lemma 1.26. *Given a free pro-p group F and an integer $n > 1$, we have $H^n(F, \mathbb{F}_p) = 0$.*

Lemma 1.27 (Inflation-restriction exact sequence)**.** *Given a group G, a normal subgroup N, and a G-module A, we have the exact sequence*

$$0 \to H^1(G/N, A^N) \to H^1(G, A) \to H^1(N, A)^{G/N} \to H^2(G/N, A^N) \to H^2(G, A).$$

Lemma 1.28. *A map $G_1 \to G_2$ of pro-p groups is surjective if and only if the corresponding map $G_1/\Phi(G_1) \to G_2/\Phi(G_2)$ is surjective.*

We now have the mechanisms necessary to relate our topological group-theoretic and group cohomological notions regarding systems of relations:

Theorem 1.29. *If G is a topologically finitely generated pro-p group, then every minimal system of relations of G has order $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p)$.*

Proof. Let

$$0 \to R \to F \to G \to 0$$

be the relevant presentation for the system E of relations in question. Lemma 1.27 yields the exact sequence

$$0 \to H^1(G, \mathbb{F}_p) \to H^1(F, \mathbb{F}_p) \to H^1(R, \mathbb{F}_p)^{F/R} \to H^2(G, \mathbb{F}_p) \to H^2(F, \mathbb{F}_p).$$

By Theorem 1.24, we have $\dim H^1(F, \mathbb{F}_p) = \dim H^1(G, \mathbb{F}_p) = d(G)$, and by Lemma 1.26 we have $H^2(F, \mathbb{F}_p) = 0$, giving us an isomorphism from $H^1(R, \mathbb{F}_p)^{F/R}$ to $H^2(G, \mathbb{F}_p)$. Thus, it suffices to prove that $|E| = \dim_{\mathbb{F}_p} H^1(R, \mathbb{F}_p)^{F/R}$. Note that by Theorem 1.18 we have $H^1(R, \mathbb{F}_p)$ isomorphic to $\mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{F}_p)$. To show that $|E| \geq \dim_{\mathbb{F}_p} H^1(R, \mathbb{F}_p)^{F/R}$, simply note that the images of the conjugates under F of the elements of E in an element of $\mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{F}_p)$ determine the entire map. Thus it suffices to show that $|E| \leq \dim_{\mathbb{F}_p} H^1(R, \mathbb{F}_p)^{F/R}$, where we may now assume that the latter is finite.

By Theorem 1.9, we have that $\mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{F}_p)$ is isomorphic to $\mathrm{Hom}_{\mathrm{cts}}(R/\Phi(R), \mathbb{F}_p)$. Now say that we have some finite minimal generating set of $H^1(R, \mathbb{F}_p)^{F/R}$. Note that each of these elements of $H^1(R, \mathbb{F}_p)^{F/R}$ must be non-zero on at least one value, so there is some set $E' \subset \mathbb{R}$ the same size as our generating set such that if an element of $H^1(R, \mathbb{F}_p)^{F/R}$ is zero on E' then it must be zero itself. Thus it suffices to show that E' is a system of relations of R. That is, letting R' be the smallest closed normal subgroup of R containing the conjugates of E' under F/R, we wish to show that $R' = R$. Note that the injection $R' \to R$ yields a surjection $f : \mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{F}_p) \to \mathrm{Hom}_{\mathrm{cts}}(R', \mathbb{F}_p)$, which we claim is bijective. To do so, consider its restriction to the surjection $\mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{F}_p)^{F/R} \to \mathrm{Hom}_{\mathrm{cts}}(R', \mathbb{F}_p)^{F/R}$. Any element of the kernel of this map must be trivial on R', and thus on E'. However, by our construction of E', this implies that our element in question is trivial. We claim that this implies a surjection $\mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{F}_p) \to \mathrm{Hom}_{\mathrm{cts}}(R', \mathbb{F}_p)$. Note that R is pro-p as it is a subgroup of a pro-p group, so if we do not have a surjection, then there is some finite subgroup R'' of R so that the map $\mathrm{Hom}_{\mathrm{cts}}(R'', \mathbb{F}_p) \to \mathrm{Hom}_{\mathrm{cts}}(R' \cap R'', \mathbb{F}_p)$ of finite vector spaces has a non-trivial kernel. This to say that the kernel of this map is non-empty, but we know that the kernel of the subgroup $\mathrm{Hom}_{\mathrm{cts}}(R'', \mathbb{F}_p)^{F/R}$ is trivial, so by Theorem 1.22 we have a contradiction, so that the map $\mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{F}_p) \to \mathrm{Hom}_{\mathrm{cts}}(R', \mathbb{F}_p)$ is in fact injective. We can write this map as $(R/\Phi(R))^* \to (R'/\Phi(R'))^*$, which is given by restriction. Thus it is clear that the map $R'/\Phi(R') \to R/\Phi(R)$ is surjective, and by Lemma 1.28 we have that the map $R' \to R$ in question is surjective, yielding our result. □

Definition 1.30. The value $r(G) = \dim H^2(G, \mathbb{F}_p)$ is known as the *relation rank* of G.

We have now laid the framework to properly describe group cohomological results regarding class field towers. In particular, we have the following

Theorem 1.31 ([Š63])**.** *For a finite group H and p a prime, let H_p be its p-part. Let G be the profinite group associated with the sequence*

$$\cdots \to \mathrm{Gal}(\mathscr{H}^2(K)/K)_p \to \mathrm{Gal}(\mathscr{H}^1(K)/K)_p \to \mathrm{Gal}(K/K)_p$$

for H topologically finitely generated, K imaginary and quadratic, and $p \neq 2$. Then $r(G) - d(G) \leq 1$.

This theorem gives us a result regarding class field towers in group cohomological terms. In fact, this information is nearly sufficient to show the existence of infinite class field towers. To see this, we introduce the main result covered in this chapter:

Theorem 1.32 (Golod-Shafarevich)**.** *If G is a non-trivial finite p-group, then $r(G) > d(G)^2/4$.*

In particular, the combinations of Theorems 1.31 and 1.32 make it clear that $d(G)$ must be at most 4. The existence of G constructed from class field towers with $d(G)$ greater than 4 (mentioned in [Zho17]) make it clear that infinite class field towers do in fact exist.

1.4 Proof of the theorem

Before we consider the theorem itself, we must prove a relevant lemma. Henceforth, we refer to $\mathbb{F}_p[G]$ as Λ.

Lemma 1.33. *Let G a finite p-group. Then for any finite G-module A with $pA = 0$, letting $b_n = \dim H^n(G, A)$ for $n \geq 0$ we have an exact sequence*

$$0 \to A \to \Lambda^{b_0} \to \Lambda^{b_1} \to \cdots .$$

such that for all n, the image of $(\Lambda^{b_n})^G$ is zero.

Proof. By Example 1.17 we have $A^G \cong (\Lambda^{b_0})^G$, as both are isomorphic to $\mathbb{F}_p^{b_0}$. We claim that we can extend this to an injection $A \to \Lambda^{b_0}$. Note that since Λ^{b_0} is an induced G-module, as it is a direct sum of induced G-modules Λ, we have the exact sequence

$$0 \to \operatorname{Hom}(A/A^G, \Lambda^{b_0}) \to \operatorname{Hom}(A, \Lambda^{b_0}) \to \operatorname{Hom}(A^G, \Lambda^{b_0}) \to 0 \tag{1.1}$$

arising from the exact sequence

$$0 \to A^G \to A \to A/A^G \to 0$$

of G-modules. Now that for a G-module A', we can impose an action of G on $\operatorname{Hom}(A', \Lambda^{b_0})$ such that for $\sigma \in G$, $a \in A'$ and $x \in \operatorname{Hom}(A', \Lambda^{b_0})$, we have $(\sigma x)(a) = \sigma x(\sigma^{-1} a)$. Under this formulation, we have Equation 1.1 as an exact sequence of G-modules, so that by Theorem 1.19 we have the long exact sequence

$$\begin{aligned} 0 &\to (\mathrm{Hom}(A/A^G, \Lambda^{b_0}))^G \to (\mathrm{Hom}(A, \Lambda^{b_0}))^G \\ &\to (\mathrm{Hom}(A^G, \Lambda^{b_0}))^G \to H^1(\mathrm{Hom}(A/A^G, \Lambda^{b_0})) \to \cdots . \end{aligned} \tag{1.2}$$

By an explicit calculation, we see for a G-module A' that $(\mathrm{Hom}(A', \Lambda^{b_0}))^G = \mathrm{Hom}_G(A', \Lambda^{b_0})$. Furthermore, a comparison of actions of G gives us

$$\mathrm{Hom}(A', \Lambda^{b_0}) = \mathrm{Hom}(A', \mathrm{Ind}_G(\mathbb{F}_p)^{b_0}) \cong \mathrm{Ind}_G(\mathrm{Hom}(A', \mathbb{F}_p^{b_0})),$$

where we impose an action of G on $\mathrm{Hom}(A', \mathbb{F}_p)$ so that for $\sigma \in G$, for a map x taking $a \in A$ to $k \in \mathbb{F}_p$, the map σx takes a to $x(a\sigma^{-1})$. Therefore, Theorem 1.21 yields that Equation 1.2 can be written as

$$0 \to \mathrm{Hom}_G(A/A^G, \Lambda^{b_0}) \to \mathrm{Hom}_G(A, \Lambda^{b_0}) \to \mathrm{Hom}_G(A^G, \Lambda^{b_0}) \to 0 \to \cdots .$$

Therefore, the map $\mathrm{Hom}_G(A, \Lambda^{b_0}) \to \mathrm{Hom}_G(A^G, \Lambda^{b_0})$ is surjective, so that our isomorphism from A^G to $(\Lambda^{b_0})^G$ extends to a map $j : A \to \Lambda^{b_0}$. Noting that $\ker(j)^G = 0$, we have from Lemma 1.22 that $\ker(j) = 0$, which is to say that j is injective. This gives for some G-module A_1 an exact sequence

$$0 \to A \to \Lambda^{b_0} \to A_1 \to 0.$$

The corresponding long exact sequence yields bijections of the form $H^n(G, A_1) \to H^{n+1}(G, A)$ for $n > 0$. Furthermore, the map $A^G \to (\Lambda^{b_0})^G$ in this sequence is bijective, so that the image of $(\Lambda^{b_0})^G$ is trivial, yielding a bijection $H^0(G, A_1) \to H^1(G, A)$. Thus by the same logic, given a G-module A_n for $n > 0$ such that $H^k(G, A_n) \cong H^{k+n}(G, A)$ for $k \geq 0$, we can construct some G-module A_{n+1} such that $H^k(G, A_{n+1}) \cong H^{k+n+1}(G, A)$ for $k \geq 0$ and such that we have an exact sequence

$$0 \to A_n \to \Lambda^{b_n} \to A_{n+1} \to 0$$

such that the image of $(\Lambda^{b_n})^G$ is trivial, as $\dim H^0(G, A_n) = \dim H^n(G, A) = b_n$. Thus we can generate G-modules A_i for $i \geq 1$ satisfying these conditions inductively, and the desired exact sequence passes through the commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& A \arrow[dr] & & & & A_2 \arrow[ur] \arrow[dr] & & & \\
& & \Lambda^{b_0} \arrow[rr] \arrow[dr] & & \Lambda^{b_1} \arrow[ur] \arrow[rr] & & \Lambda^{b_2} \arrow[rr] \arrow[dr] & & \cdots . \\
& & & A_1 \arrow[ur] \arrow[dr] & & & & A_3 \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

□

Proof of Theorem 1.32. For a G-module A such that $pA = 0$, we define for some m the series

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_m = A$$

where $A_0 = 0$ and $A_{n+1}/A_k = (A/A_n)^G$ for $n \geq 0$. By Lemma 1.22, we have that $A_n \neq A_{n+1}$ so long as $A_n \neq A$. Furthermore, by induction an injective G-homomorphism $h : A \to B$ will satisfy $A_n = h^{-1}(B_n)$ for all $n \geq 0$. Letting $s_n(A) = \dim(A_{n+1})$, we define the *Poincaré polynomial* of A

$$P_A(t) = \sum_{n \geq 0} \dim(A_{n+1}/A_n),$$

which has the property that if $0 < t < 1$ then

$$\frac{P_A(t)}{1-t} = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \dim(A_{n+1}/A_n) \right) t^n = \sum_{n \geq 0} s_n(A) t^n.$$

Let $r = r(G)$ and $d = d(G)$. Letting $A = \mathbb{F}_p = \Lambda_1$, quotienting a copy of A out of the exact sequence given by Lemma 1.33 yields the exact sequence

$$0 \to E \to D \to R \tag{1.3}$$

where $E = \Lambda/\Lambda_1$, $D = \Lambda^d$, $R = \Lambda^d$, and the image of D_1 is 0. By induction, the image of D_{n+1} is a subset of R_n for $n \geq 0$, and since $E \to D$ is injective, we have already shown that the image of E_{n+1} is contained in D_{n+1} for $n \geq 0$. Thus for $n \geq 0$ we can restrict Equation 1.3 to the sequence

$$0 \to E_{n+1} \to D_{n+1} \to R_n,$$

so that if we let $s_{-1}(R) = 0$ then for $n \geq 0$ we have

$$s_n(D) \leq s_n(E) + s_{n-1}(R),$$

so that for $0 < t < 1$ we have

$$\frac{P_D(t)}{1-t} \leq \frac{P_E(t)}{1-t} + \frac{tP_R(t)}{1-t}.$$

However, letting $P(t) = P_\Lambda(t)$ we have

$$\begin{aligned} P_E(t) &= \frac{P(t)-1}{t}, \\ P_D(t) &= dP(t), \\ P_R(t) &= rP(t). \end{aligned}$$

Therefore for $0 < 1 < t$ we have

$$\begin{aligned} dP(t) &\leq \frac{P(t)-1}{t} + trP(t) \\ 1 &\leq (rt^2 - dt + 1)P(t). \end{aligned}$$

Since $P(t)$ has positive integer coefficients, degree at least 1, and a positive integer constant term, we have for $0 < t < 1$ that $rt^2 - dt + 1 > 0$. Thus it suffices to show that $0 < d/2r < 1$, as this would imply $r > d^2/4$, as desired. To show this, note that the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_p \to 0$$

yields by Theorem 1.19 a long exact sequence with subsequence

$$H^1(G, \mathbb{Z}) \to H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{Z}) \to H^2(G, \mathbb{Z}) \to H^2(G, \mathbb{F}_p).$$

Now by Example 1.17, we have $H^1(G, \mathbb{Z}) = 0$. Note that each of these groups is finitely generated and abelian, where $\dim H^1(G, \mathbb{F}_p) = d$ and $\dim H^2(G, \mathbb{F}_p) = r$. Letting $k = \dim H^2(G, \mathbb{Z})$, the number of generators in the first copy of $H^2(G, \mathbb{Z})$ after quotienting out its copy of $H^1(G, \mathbb{F}_p)$ is $k - d$. Thus the quotient of the second copy of $H^2(G, \mathbb{Z})$ with this quotient has d generators, so since this is injected into $H^2(G, \mathbb{F}_p)$, we have $d \leq r$. However, since G is non-trivial, we have $d > 0$ by its topological definition, Theorem 1.8, and Corollary 1.10, so that $0 < d/2r < 1$, as desired. $\square$

Chapter 2

4-Manifolds and Number Fields

While the Golod-Shafarevich problem proves that not all class field towers stabilize, the question of classifying which number fields correspond to stabilizing class field towers is still largely open. Nikolaev proposes in [Nik21] a method for extracting a real embedding of a number field from a compact 4-dimensional manifold, which may allow us to understand class field towers via topological machinery. We provide an overview of the topics relevant to this proposal, along with its motivations and mechanisms, and attempt to provide a refinement which may lend itself to a more rigorous understanding of the underlying concepts. It should be noted however that much of the work regarding class field towers is not yet in a position to be properly inspected, so this chapter will not say much about class field towers themselves, and will instead focus primarily on the construction of the mechanism designed to produce a sequence of number fields. Sections 2.2 and 2.3 pull largely from [Bla86] and [Dix77].

2.1 Diffeomorphism groups

Definition 2.1. Given a smooth n-manifold X, the group $\mathit{Diff}(X)$ is the group of orientation-preserving diffeomorphic automorphisms of X. Furthermore, given a smooth orientation-preserving embedding $x : \mathbb{D}^n \to X$, the group $\mathit{Diff}(X, x)$ is the subset of $\mathit{Diff}(X, x)$ which fixes $\operatorname{im} x$.

Now there are many topologies that can be applied to $\mathit{Diff}(X)$ to give it a topological group structure, such as the *compact-open topology* ([Hir76], Section 2.1) or the *Whitney C^0-topology* ([GG73], Definition 2.3.1). We will not take on any particular topology here, but we will assume in this section that our topology is locally connected on both $\mathit{Diff}(X)$ and $\mathit{Diff}(X, x)$. Let $\mathit{Diff}_0(X)$ and $\mathit{Diff}_0(X, x)$ be the connected components of $\mathit{Diff}(X)$ and $\mathit{Diff}(X, x)$ containing the identity, respectively. Note that a space is locally connected if and only if each component is connected, so a condition that our topology is locally connected would be necessary and sufficient to ensure that $\mathit{Diff}(X)/\mathit{Diff}_0(X)$ and $\mathit{Diff}(X,x)/\mathit{Diff}_0(X,x)$ are discrete.

2.2 C^*-algebras

Definition 2.2. A *Banach algebra* A is an associative algebra over $\mathbb{R}$ or $\mathbb{C}$ with a norm $\|\cdot\|$ such that A is closed under the corresponding metric and $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

Definition 2.3. An *involutive algebra* is an algebra A with an involution $\cdot^*$ on A such that for all $x \in A$ and $\lambda \in \mathbb{C}$ we have

- $(x^*)^* = x$,
- $(x+y)^* = x^* + y^*$,
- $(xy)^* = x^* y^*$,
- $(\lambda x)^* = \overline{\lambda} x^*$.

Definition 2.4. A *C^*-algebra* is an involutive Banach algebra A over $\mathbb{C}$ such that for all $x \in A$ we have $\|x^* x\| = \|x\| \cdot \|x^*\|$.

Definition 2.5. The C^*-algebra M_n is the algebra of n-by-n matrices with entries in $\mathbb{C}$, with norm given by taking the square root of the largest absolute value of an eigenvalue and involution given by taking the conjugate transpose.

Theorem 2.6 ([Put], Theorem 1.7.1)**.** *Any finite-dimensional C^*-algebra is expressible for some finite multiset S of positive integers as a direct sum $\bigoplus_{s \in S} M_s$.*

Definition 2.7. A measure μ on a topological group G is *left-invariant* if for $g \in G$ and some subset S on which μ is defined we have $\mu(gS) = \mu(S)$ where S is a measurable set.

Theorem 2.8. *([Amb47]) Given a locally compact group G, there exists a non-trivial left-invariant measure of G, known as the* Haar measure *of G, which is unique up to a constant, which assigns to every compact subset of G some finite measure, and which is defined only on the σ-algebra generated by the compact subsets of G and the subsets of compact subsets which are assigned a measure of zero. This measure is non-zero on all measurable open sets.*

Given the Haar measure μ of a group G, we may define the space of $L^1(G)$ functions of G as space of measurable functions f on G such that $\int |f| d\mu$ is finite. Furthermore, we may give $L^1(G)$ a norm $\|\cdot\|$ and an involution $\cdot^*$ so that for some $f \in L^1(G)$ we have $\|f\| = \int |f| d\mu$ and for all $g \in G$ we have $f^*(g) = \overline{f(g^{-1})}$.

Definition 2.9. Given an algebra A with a norm $\|\cdot\|$, an *approximate identity* is a set $\{e_\lambda \mid \lambda \in \Lambda\} \subset A$ for some directed set Λ with relation $\leq$ such that each e_i satisfies $\|e_i\| \leq 1$ and for each $a \in A$ and $\varepsilon > 0$ there is some $\lambda \in \Lambda$ such that if $\lambda \leq \lambda'$ then $\|ae_{\lambda'} - a\|$ and $\|e'_\lambda a - a\|$ are less than ε.

Theorem 2.10 ([Dix77], Result 13.2.5)**.** *Given a locally compact group G, the normed algebra $L^1(G)$ has an approximate identity.*

Definition 2.11. Given a Hilbert space H, we have an algebra $\mathscr{L}(H)$ of continuous endomorphisms of H, on which we can define an involution via the adjoint ([BN00], Theorem 11.4 and Definition 17.1).

Definition 2.12. If A is an involutive algebra, a *representation* of A is a map π from A to $\mathscr{L}(H)$ for some Hilbert space H which preserves addition, multiplication, scalar multiplication and involution.

Definition 2.13. Given an involutive Banach algebra A with approximate identity and with R its set of representations, we can define a map $\|\cdot\|'_A$ on A, written as $\|\cdot\|'$ when A is clear, such that for $x \in A$ we have

$$\|x\|' = \sup_{\pi \in R} \|\pi(x)\|.$$

Theorem 2.14 ([Dix77], Proposition 2.7.1)**.** *Given A as in the above, for all $x, y \in A$ and $\lambda \in \mathbb{C}$ the map $\|\cdot\|'$ will satisfy*

- $\|x\|' \leq \|x\|$,
- $\|\lambda x\|' = |\lambda| \cdot \|x\|'$,
- $\|x + y\|' \leq \|x\|' + \|y\|'$,
- $\|xy\|' \leq \|x\|' \cdot \|y\|'$,
- $\|x^*\|' = \|x\|'$,
- $\|x^* x\|' = \|x\|'^2$.

Theorem 2.15 ([Dix77] Result 2.7.2)**.** *Given A as in the above, and letting $I(A)$ be the set of $x \in A$ such that $\|x\|' = 0$, the map $\|\cdot\|'$ yields a norm of the quotient space $A/I(A)$. This quotient satisfies all the definitions of a C^*-algebra, with the possible exception of completeness.*

Definition 2.16. The C^*-algebra achieved by completing $L^1(G)/I(L^1(G))$ under the metric $\|\cdot\|'$ is denoted $C^*(G)$.

Note that if G is discrete, the counting measure satisfies the properties of the Haar measure. Thus in this case $L^1(G)$ is naturally isomorphic to the group algebra $\mathbb{C}[G]$ with the involution taking λg to $\overline{\lambda} g^{-1}$ for $g \in G$ and $\lambda \in \mathbb{C}$.

Theorem 2.17. *If G is finite, the group algebra $\mathbb{C}[G]$ with a norm taking sums of absolute values of coordinates and an involution taking λg to $\overline{\lambda} g^{-1}$ for $g \in G$ and $\lambda \in \mathbb{C}$ is a C^*-algebra.*

Proof. It is clear that this norm yields an involutive Banach algebra, so it suffices to note that for any $x \in A$ we have $\|x^*x\| = \|x\| \cdot \|x^*\|$. For $\sum_{g\in G} a_g g \in \mathbb{C}[G]$ we have

$$\begin{aligned}
\left\| \left(\sum_{g\in G} a_g g\right)^* \left(\sum_{g\in G} a_g g\right) \right\| &= \left\| \sum_{g,h\in G} a_g \overline{a_h} g h^{-1} \right\| \\
&= \left\| \sum_{g\in G} g \sum_{h\in G} a_{gh} \overline{a_{h^{-1}}} \right\| \\
&= \sum_{g,h\in G} |a_g| \cdot |a_h| \\
&= \left(\sum_{g\in G} |a_g| \right)^2 \\
&= \left\| \sum_{g\in G} a_g g \right\| \left\| \left(\sum_{g\in G} a_g g \right)^* \right\|.
\end{aligned}$$

□

Theorem 2.18 ([Dix77], Result 2.7.3)**.** *If $L^1(G)$ is a C^*-algebra, it is naturally identified with its enveloping C^*-algebra.*

Now note that if some closed subgroup H of G contains some measurable open subset of G, then the restriction of the Haar measure of G to H will also define a Haar measure, so that $L^1(H)$ will have a natural inclusion into $L^1(G)$. For finite G, this gives us a natural inclusion $C^*(H) \to C^*(G)$ by Theorem 2.18.

Note that if we do have a natural inclusion from $L^1(H)$ to $L^1(G)$, then every representation of $L^1(G)$ induces a representation of $L^1(H)$, so that the map $\|\cdot\|'_{L^1(G)}$ will be bounded above by $\|\cdot\|'_{L^1(H)}$, yielding a map $L^1(H)/I(L^1(H)) \to L^1(G)/I(L^1(G))$. In fact, this gives us a map of Cauchy sequences as well, so that we have a map from $C^*(H)$ to $C^*(G)$. However, it is not clear in general that this will be an inclusion.

We wish to now consider a set of C^*-algebras which are more closely related to linear-algebraic structures. Consider the following

Definition 2.19. An *approximately finite-dimensional algebra*, or *AF algebra*, is a direct limit of finite-dimensional C^*-algebras.

We also survey the conditions on which $C^*(G)$ is known to be AF or not.

Theorem 2.20 ([Bla86], Corollary 11.1.2)**.** *If G is compact, then $C^*(G)$ is AF.*

Now in general, a set of necessary and sufficient conditions to determine whether a locally compact group begets an AF algebra is unknown. However, we do point out some conditions which illuminate that we will not be able to assume that C^* is AF in general.

Definition 2.21. Given a Banach algebra A, we construct a Banach algebra A^+ with an identity by giving it elements (a, λ) for $a \in A$ and $\lambda \in \mathbb{C}$, so that addition is component-wise and for $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$ we have $(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$.

Definition 2.22. For a unital C^*-algebra A, the *real rank* $RR(A)$ is the smallest integer r such that if $n \leq r + 1$ for any n-tuple $(x_1, \ldots, x_n)$ of elements which are their own involution and any $\varepsilon > 0$, there is some n-tuple $(y_1, \ldots, y_n)$ of elements which are their own involution such that $\sum_i y_i^2$ is invertible and

$$\left\| \sum_i (x_i - y_i)^2 \right\| < \varepsilon.$$

If A is not unital, then the real rank of A is that of A^+.

Theorem 2.23. *Finite-dimensional C^*-algebras have real rank zero.*

Proof. If the relevant matrix x has determinant other than 0, we can set $y = x$. Otherwise, we can add an arbitrarily small multiple of the identity matrix to get y, so that y has nonzero determinant and $y - x$ will be an arbitrarily small multiple of the identity. □

Theorem 2.24 ([BP91], Theorem 3.1)**.** *Inductive limits of C^*-algebras of real rank zero also have real rank zero themselves.*

Corollary 2.25. *AF algebras have real rank zero.*

Proof. This follows from Theorems 2.23 and 2.25. $\square$

Theorem 2.26 ([Kan93], Lemma 2)**.** *If G is a torsion-free discrete abelian group, then the real rank of G is positive.*

Thus we have that not all discrete groups have real rank zero, so that by Corollary 2.25 not every discrete group will yield an AF algebra.

2.3 Stationary AF algebras and dimension groups

Before we continue, we provide information as to the structure of an AF algebra. In particular, say that we have a map $\phi : A \to B$ of finite-dimensional C^*-algebras such that $A = \bigoplus_{i=1}^{n_1} M_{k_{1,i}}$ and $B = \bigoplus_{i=1}^{n_2} M_{k_{2,i}}$ for some integers $n_1, n_2, k_{1,i_1}, k_{2,i_2}$ for $1 \leq i_1 \leq n_1$, $1 \leq i_2 \leq n_2$. Now for a finite sequence of matrices $\{a_i\}$ for $1 \leq i \leq m$ for some positive integer m, let

$$\operatorname{diag}(a_1, \ldots, a_m) = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_m \end{bmatrix}.$$

Then as is noted in [Bla86] Section 7.2 there is some matrix $M_\phi = \{m_{ij}\}$ with non-negative integer entries such that up to a permutation in each $M_{k_{2,i}}$, the map $A \to M_{k_{2,i}}$ is the map taking $(a_{1,1}, \ldots, a_{1,n_1})$ for $a_{1,j} \in M_{k_{1,j}}$ to $\operatorname{diag}(a_{1,1}, \ldots, a_{1,2}, \ldots, a_{1,n_1}, 0, \ldots)$, where each $a_{1,j}$ is repeated m_{ij} times. Note that for all i the sum of terms in row i must be at most $k_{2,i}$. This map is an inclusion if and only if each column of M is non-empty, and the identity of A maps to the identity of B if and only if for all i the sum of the terms in row i is $k_{2,i}$. Thus an AF algebra $\bigcup_i A_i$ with maps $\phi_i : A_i \to A_{i+1}$ for finite-dimensional C^*-algebras A_i has an associated sequence of matrices M_{ϕ_i} with non-negative integer entries. This sequence will not necessarily be unique: note that for some given sequence, we can combine ϕ_i with ϕ_{i+1} to get a map from A_i to A_{i+2} with corresponding matrix $M_{\phi_{i+1}} M_{\phi_i}$.

Definition 2.27 ([Nik17], Section 3.5.2)**.** If an AF algebra $\bigcup A_i$ for finite-dimensional C^*-algebras A_i has an associated sequence of matrices M_{ϕ_i} which are all equal, have each entry strictly positive, and are in $\mathrm{GL}_n(\mathbb{Z})$, then $\bigcup A_i$ is said to be *stationary*.

The following theorem will prove to be useful in the context of extracting number fields from stationary AF algebras:

Theorem 2.28. *(Perron-Frobenius) A square matrix with positive entries has a unique eigenvalue which is positive and larger than all other eigenvalues in absolute value. Its eigenvector has positive entries. Such an eigenvalue is called Perron.*

Proof. See [Mac00]. ☐

We now turn our attention to another object known as K_0 of an arbitrary C^*-algebra, from which our number field will be extracted.

For a C^*-algebra A we let $M_i(A)$ denote the algebra of i-by-i matrices with entries in A, and let $M_\infty(A)$ be the natural limit by inclusion of the algebras $M_i(A)$. We wish to create a group from the idempotents of $M_\infty(A)$. We let two idempotents p, q be equivalent if for some $x, y \in A$ we have $xy = p$ and $yx = q$.

Definition 2.29 ([Bla86], Definitions 5.1.2 and 5.3.1)**.** Given a Banach algebra A, the monoid of equivalence classes of idempotents of $M_\infty(A)$, such that for idempotents a, b we define their sum as $\mathrm{diag}(a, b)$, is written $V(A)$. The group constructed from this monoid by taking formal inverses is written $K_{00}(A)$.

It is clear that a map $\phi : A \to B$ of C^*-algebras induces a homomorphism $\pi_* : V(A) \to V(B)$.

Definition 2.30. Let A be a C^* algebra. Then the *dimension group* $K_0(A)$ of A is the kernel of the map $\phi_* : K_{00}(A^+) \to K_{00}(\mathbb{C})$ given by the map $\pi : A^+ \to \mathbb{C}$ which quotients out A.

Theorem 2.31 ([Bla86], Theorem 5.5.5)**.** *If $M_\infty(A)$ contains an approximate identity consisting of idempotents, then the canonical map from $K_{00}(A)$ to $K_0(A)$ is an isomorphism.*

Note that M_n contains an identity for positive integers n, so $K_0(M_n)$ is generated by the projections of M_n. Example IV.2.1 of [Dav96] points out that $K_0(M_n)$ is isomorphic to $\mathbb{Z}$, generated by a projection in M_n of rank one, so that a finite-dimensional C^*-algebra $A = \bigoplus_{s \in S} M_s$ satisfies $K_0(A) \cong \mathbb{Z}^{|S|}$. Furthermore, an AF-algebra A of the form $\bigcup A_i$ for finite-dimensional A_i has an approximate identity $\{e_j\}$ of $M_\infty(A)$ such that e_j is the j-by-j matrix consisting of the identity of A_j.

Now let $K_0(A)_+$ refer to the image of $V(A)$ in $K_0(A)$. Now a map $A \to B$ of finite-dimensional C^*-algebras yields a map $K_0(A) \to K_0(B)$. In particular, in a stationary AF algebra A given by an n-by-n matrix M, we will have a sequence of inclusions of the dimension groups of the finite-dimensional C^*-algebras A_i which comprise A of the form

$$\mathbb{Z}^n \xrightarrow{M} \mathbb{Z}^n \xrightarrow{M} \mathbb{Z}^n \to \cdots ,$$

where each of these has the non-negative terms as its corresponding $K_0(A_i)_+$. Since M is invertible, this sequences is one of automorphisms, and $K_0(A) \cong \mathbb{Z}^n$. Since M must have some Perron eigenvalue, an element $K_0(A)$ will be in $K_0(A)_+$ if the corresponding element of $K_0(A_1)$ has positive dot product with the corresponding eigenvector. If the dot product at most zero but the element is not zero, then the element will remain on or below the normal of the eigenvector and avoid zero, so since the eigenvector has positive entries, such an element of $K_0(A)$ will not be in $K_0(A)$. Thus letting $K_0(A)_+ \setminus \{0\}$ be the set of positive elements, we have some linear ordering of $\mathbb{Z}^n$, which we can map into $\mathbb{R}$, positing further that 1 is contained in the image of this map. We now present the following

Theorem 2.32 ([Han81], Theorem II)**.** *For n a positive integer and G the image of a linear ordering of $\mathbb{Z}^n$ into $\mathbb{R}$ given by some stationary AF algebra, the tensor $G \otimes \mathbb{Q}$ is a number field. Conversely, every real embedding of a number field can be expressed as $G \otimes \mathbb{Q}$ for some such G.*

Thus we have a method for constructing a number field from a stationary AF algebra such that any real embedding of a number field can be constructed.

2.4 Stable diffeomorphism groups

Let X and Y be n-manifolds, let x be an embedding of an n-disc in X, and let y and y' be embeddings of n-discs in Y. Given X, x, Y, and y allows us to construct a connected sum $X\#Y$ with a smooth structure which preserves orientations and the smooth structure on the canonical copies of $X \setminus x$ and $Y \setminus y$ lying in this direct sum ([Kup], Definition 9.2.1). Furthermore, this gives us an injection $\mathit{Diff}(X, x) \to \mathit{Diff}(X\#Y, y')$, where we let y' refer to the image of y' in the map $Y \to X\#Y$. We can consider then a sequence

$$\mathit{Diff}(X, x) \to \mathit{Diff}(X\#Y, y') \to \mathit{Diff}(X\#Y\#Y, y'') \to \cdots$$

of diffeomorphism groups, where the connected sums bring with them embeddings $y', y'', \ldots$ of n-discs. Letting the *stable diffeomorphism group* of X with respect to Y be the direct limit of this sequence, we have the following

Theorem 2.33 ([Szy08], Proposition 3)**.** *If two complex algebraic surfaces are birationally equivalent, then they share a stable diffeomorphism group with respect to the reversed-orientation copy* $\overline{\mathbb{C}P^2}$ *of the complex projective plane.*

This fact relies on the fact ([GS99], Definition 2.2.7) that the blowup of a complex algebraic surface X is $X\#\overline{\mathbb{C}P^2}$. In the opposite direction, a finite number of blowdowns will result in a *minimal model*, which is a complex algebraic surface X such that any rational map $f : X \to X'$ for X' also a complex algebraic surface is an isomorphism (see [GS99], Section 2.2 and [Har77], Section V.5; note also that [Har77] uses the term *relatively minimal model* in place of *minimal model*).

2.5 Nikolaev's proposal

Nikolaev's paper seeks to relate a 4-manifold with the number field associated to the dimension group of the C^*-algebra of $\mathit{Diff}(X)/\mathit{Diff}_0(X)$, given a topological structure constructed by Etesi in [Ete16]. The idea is to consider structures X, Y such that we have an inclusion $\mathit{Diff}(X)/\mathit{Diff}_0(X) \to \mathit{Diff}(X\#Y)/\mathit{Diff}_0(X\#Y)$ of stationary AF algebras and the map of number fields

$$K_0(C^*(\mathit{Diff}(X)/\mathit{Diff}_0(X))) \to K_0(C^*(\mathit{Diff}(X\#Y)/\mathit{Diff}_0(X\#Y)))$$

is an inclusion from a field to its Hilbert class field. Our analysis of this focuses less on the number field aspect, but rather on the construction of this map.

First, we consider the conditions on Etesi's topological structure which must be shown in order for this mapping to not only be well-defined but also to yield an injection. First, recall that a natural map is only given from $\mathit{Diff}(X, x)$ to $\mathit{Diff}(X\#Y, y')$, so this map must be completely representative of $\mathit{Diff}(X, x)$: that is, every diffeomorphism must share a connected component with one which fixes x. Furthermore, in order for our map of quotients to be well-defined, the image of $\mathit{Diff}_0(X)$ must be a subset of $\mathit{Diff}_0(X\#Y)$. Finally, in order for our map to be an inclusion, this image must be the entire intersection $\mathit{Diff}(X) \cap \mathit{Diff}_0(X\#Y)$.

As intuition, say for the moment that there exists a topology on $\mathit{Diff}(X)$ whose connected component containing the identity consists exactly of the isotopies of X. The *isotopy extension theorem* ([Hir76], Theorem 8.1.3) states that any isotopy of the disc in X extends to an isotopy of X. Furthermore, it is shown in Corollary 1 of [Pal60] that any two orientation-preserving embeddings of a disc are isotopic, so that any $f \in \mathit{Diff}(X)$ can construct an isotopy to an element of $\mathit{Diff}(X, x)$ by extending an isotopy from x to the image of x under f. Furthermore, any isotopy of $\mathit{Diff}(X, x)$

certainly maps to an isotopy of *Diff*$(X\#Y, y')$. In this case, the only criterion which is not clearly met is that of injectivity. In particular, injectivity would imply that if two diffeomorphisms were not isotopic in X, then their images would not be isotopic in $X\#Y$, which is not clear. Thus in order to provide viability to this process, one would first need to show that this map was injective under Etesi's topological structure, or that the kernel became irrelevant elsewhere in the process of constructing this map of number fields.

Furthermore, Nikolaev's proof assumes the inclusion of the C^*-algebras arising from these groups from the assumption that they are discrete. This creates two issues. First, this implies that Etesi's topological structure is locally connected, which must also be shown. Second, in order to realize K_0 of a C^*-algebra as a number field, it must be stationary AF. Nikolaev mistakenly references Theorem 2.20 as only requiring local compactness. However, as it in fact requires compactness, this property will not be satisfied unless our group is finite. However, in this case, by Theorems 2.17 and 2.18 our AF algebra will also be finite-dimensional, and our dimension group will be a finite-dimensional vector space whose non-negative elements are simply those with non-negative entries.

One possible solution would be to take the profinite completion. Note that the profinite completion of *Diff*(X) would trivialize *Diff*$_0(X)$: if not, then some open normal subgroup of *Diff*(X) of finite index would not contain all of *Diff*$_0(X)$, so that its corresponding cosets would yield a non-trivial partition of *Diff*$_0(X)$ into open sets, contradicting its definition. Thus we can take the profinite completion of *Diff*(X) directly. Furthermore, Ribes and Zalesskii show in Section 3.2 of [RZ10] how to construct from an inclusion $H \to G$ of groups H, G with profinite completions $\hat{H}$, $\hat{G}$ a map $\hat{H} \to \hat{G}$, and show that this map is an inclusion if and only if the topology induced on $\hat{H}$ by $\hat{G}$ is the given topology of $\hat{H}$ ([RZ10], Lemma 3.2.6). We also recall that our map of C^*-algebras resulting from an inclusion of groups was reliant on the subgroup being closed and containing an open subset of the group itself, and its inclusion was only shown for finite groups. Thus in order to justify the map here, we would not only need to show that our initial map constituted an injection and show that the resulting map of profinite completions comprised an injection as well, but also that the resulting map of C^*-algebras was well-defined and injective. In fact, even an inclusion of C^*-algebras does not imply an inclusion of the corresponding dimension groups. For example, the map from M_1^2 to M_2 which maps corresponding 2-by-2 matrices yields a map $\mathbb{Z}^2 \to \mathbb{Z}$ of dimension groups. Finally, we must show that these resulting C^*-algebras are stationary AF, rather than merely AF, so that K_0 of these groups do in fact yield number fields.

While the primary claim of Nikolaev's paper relies on class field towers, much of the machinery is not quite built to solve this problem at present. The proposed approach looks to unite the sequence associated to the stable diffeomorphism group of compact 4-manifolds with a class field tower, by taking K_0 of each term. In particular, Nikolaev claims that by considering minimal models, our maps will eventually consist solely of isomorphisms, so that the proposed corresponding class field tower could be shown to stabilize. However, the given direction of the maps deals with maps to blowups rather than blowdowns, going in the opposite direction of that which is used to obtain a minimal model. Furthermore, it is not clear which number fields could be realized as dimension groups of stationary AF algebras of 4-manifolds in this way. Every number field in question must have at least one real embedding, namely the embedding given by the corresponding dimension group, and we have noted that every such number field can be obtained from some stationary AF algebra, but it is unclear that all such stationary AF algebras can be obtained from 4-manifolds via the method described. However, if these issues can be resolved, it may in fact still be possible to use this process or construct a similar one to extract information about number fields using information about complex algebraic surfaces or 4-manifolds in general.

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