Visually Ordering Ideals by Inclusion

Michael Gintz

For any UFD, we can categorize its elements by the number of prime factors of its irreducible representation, counting multiplicity. To visually represent this, we can place the elements of our ring (up to units) on different vertical lines, as shown in Figure 1 for the ring of integers. As is canon we place 1 on a line corresponding to 0 and 0 on a line corresponding to $-\infty$. Then the signed heights of points are additive under multiplication. This structure can be thought of as an analog to a number line, where coordinates are additive under addition.

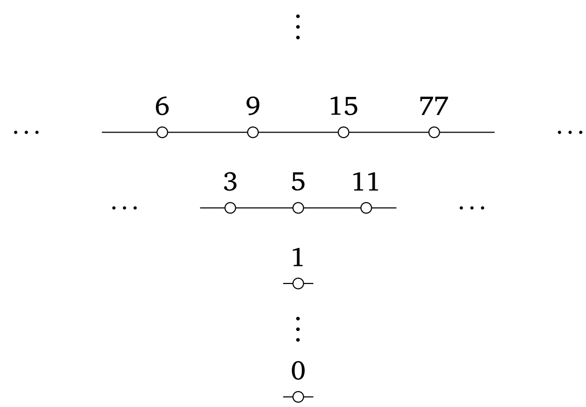

Figure 1: A "vertical number line" for the integers.

We wish to integrate ideals into this drawing as well. Consider first a principal ideal, say $\langle 2 \rangle$. We will need to reposition our elements along our horizontal lines, but with the exception of 0 we can place all of the points within an angular region centered at 2:

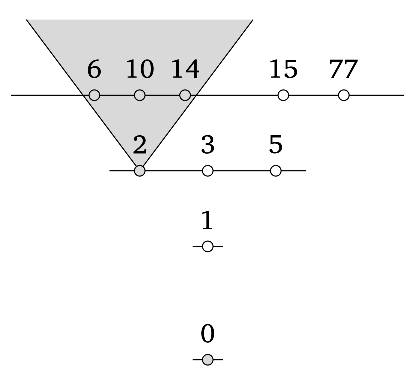

Figure 2: Drawing in the principal ideal $\langle 2 \rangle$.

After only drawing in this one ideal, any element that we wish to draw in has a place in our diagram. However, consider Figure 3, in which we have attempted to shade in the union of the ideals $\langle 2 \rangle, \langle 3 \rangle$, and $\langle 5 \rangle$:

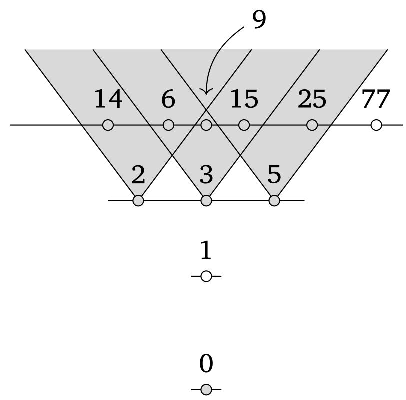

Figure 3: With too many ideals, it becomes impossible to give a location to some elements.

Note that in this drawing there is no space to place the element 10, which is contained in $\langle 2 \rangle$ and $\langle 5 \rangle$ but not $\langle 3 \rangle$. To remedy this, we can consider this drawing as a projection from a many-dimensional space into a 2-dimensional space, such as in Figure 4.

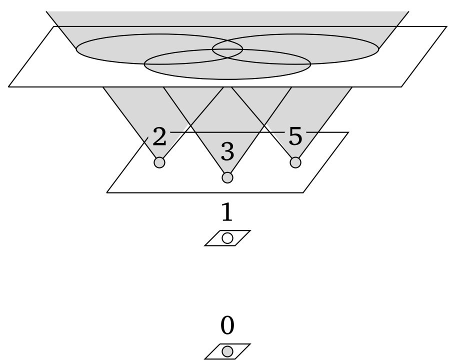

Figure 4: A many-dimensional interpretation of our diagram.

Of course, once we then project our image into two dimensions we introduce some ambiguity into our diagram, for the same reason that made Figure 3 infeasible. Since this ambiguity only arises when we add in more than two ideals, we can only draw in lines which separate two regions at a time, such as in Figure 5. Now in Figure 5 we have also removed our horizontal lines and placed zero at the top of our diagram. While the horizontal lines may still be useful in some problems involving UFDs, this points out that this diagram can actually be given for arbitrary rings. Then, placing 0 at the top of our diagram makes more sense, as it is in the intersection of every ideal. Indeed, if we gave 0 a height of ∞ in our general UFD, we see that additivity of height under multiplication is still preserved. Placing zero at the top also removes the need to show whether points on our diagram are filled in, as their existence in various shaded regions will be determined by their location on the diagram.

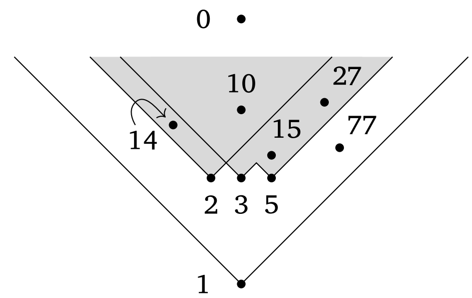

Figure 5: Properly projecting our many-dimensional diagram.

Now that we have provided visual cues for principal ideals, we can provide some for other ideals as well. For ideals with a given set of generators, we can provide a flat edge, indicated either by a finite set of generators or the name of the generating set. For general ideals, we can simply provide an asterisk.

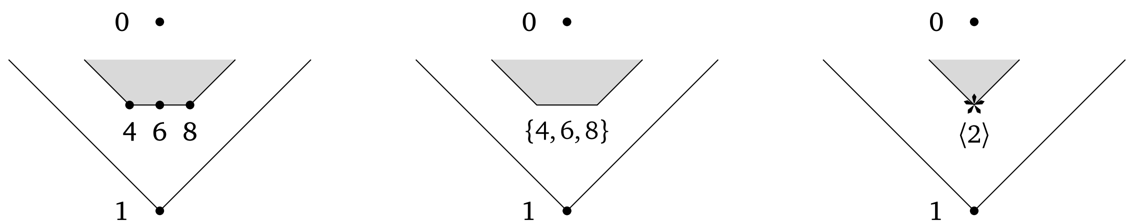

Figure 6: A few more visual cues for ideals.

Mumford's Treasure Map

Mumford's treasure map (Figure 7) is a well-known pictoral representation of the Zariski topology of $\mathbb{Z}[X]$, seen first in [Mum99] (though its name comes from [LBr08]). The points of our topology are represented by points of varying sizes. The largest point is the "generic point" corresponding to the prime ideal $\langle 0 \rangle$. Each generic point corresponds to some shape in the drawing: the generic point for $\langle 0 \rangle$ corresponds to the big rectangle, and the generic points along the edges, which represent the other principal prime ideals, correspond to the lines they lie on. In general, these shapes are built to contain the points of all of the primes containing the prime represented by the generic point. For example, since the shape corresponding to $\langle 0 \rangle$ contains all of the points, we are being shown that every prime ideal contains 0. We line up the ideals generated by prime numbers along the top, and ideals generated by irreducible polynomials along the side. Le Bruyn provides and posits plenty of useful insight that can be extracted from this drawing, but for our purposes we only need the information we have already given, and the fact that it is useful.

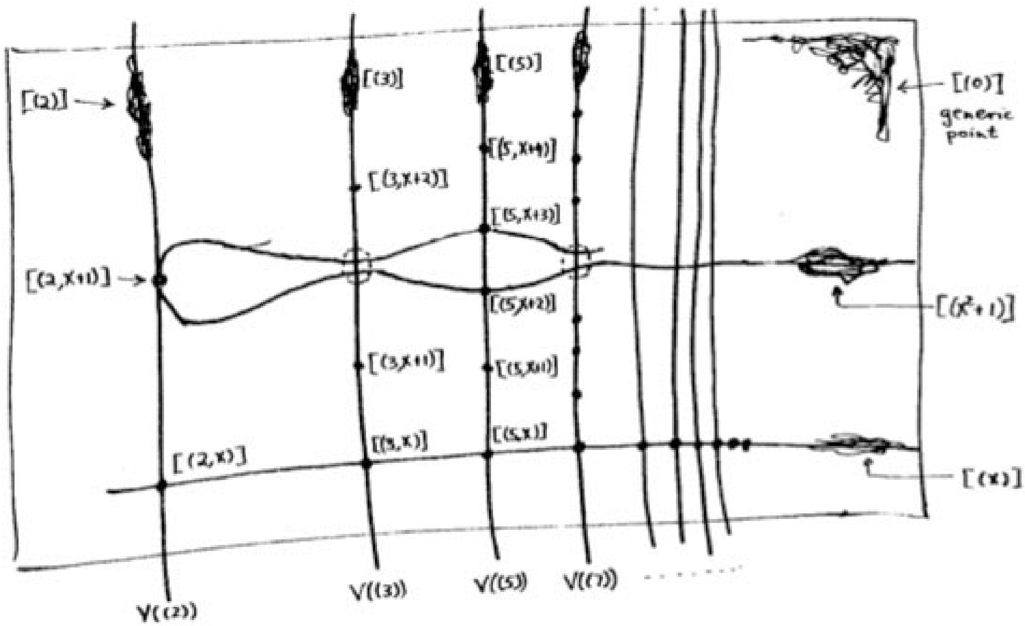

Figure 7: Mumford's treasure map.

We mention this image because we claim that it follows naturally from our ideal diagrams on general rings. First, recall that the prime ideals of $\mathbb{Z}[X]$ are of the form $\langle 0\rangle, \langle p\rangle, \langle f(X)\rangle$, or $\langle f(X), p\rangle$, where p is prime and $f(X)$ is irreducible, and when paired with a prime p it must be irreducible in $\mathbb{F}_p[X]$ as well. Now in terms of inclusion, we have a rough idea of where to place our ideals: in particular, we should place the $\langle f(X), p\rangle$ ideals closest to the bottom as they will contain other ideals, and the $\langle p\rangle$ and $\langle f(X)\rangle$ ideals should be placed above that, with 0 at the top as always. However, we do not yet draw in any lines to represent inclusion: recall that with more than 2 our diagram begins to lose information by necessity. This information is given in Figure 8.

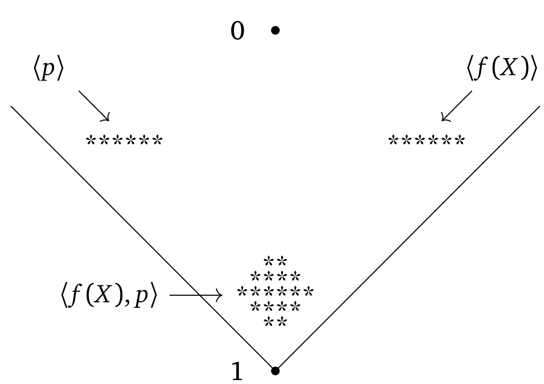

Figure 8: Roughly placing our prime ideals on an ideal diagram.

Now since it is impossible to draw our ideals in as triangles all at once, we resort to drawing lines to represent inclusion. In particular, we will draw a line from a prime ideal to all of the prime ideals which contain it. We will omit lines for 0: its location at the top of the diagram ought to be sufficient for us to recognize that all prime ideals contain it. Some lines will have to be threaded back, so as to account for all inclusions. The result is given in Figure 9:

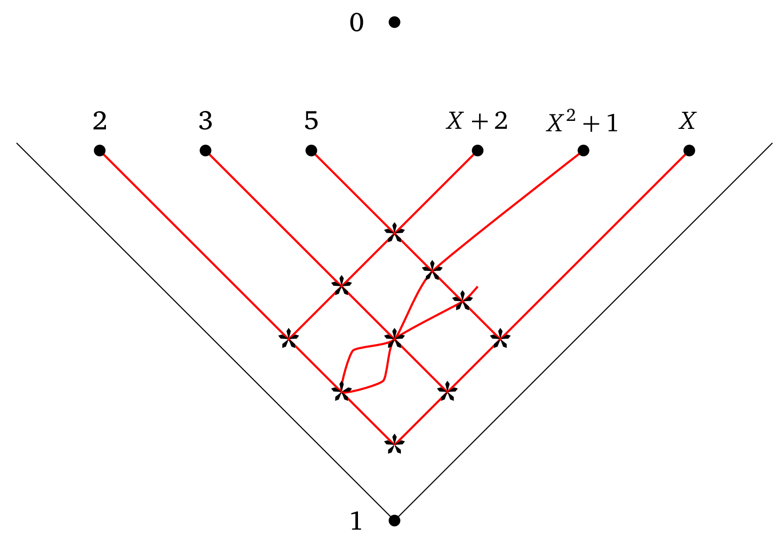

Figure 9: Realizing Mumford's treasure map with our ideal diagrams.

Now it is clear that Figure 9 is effectively identical to the treasure map. Of course, to say that Mumford's treasure map follows from these ideal diagrams is extremely dishonest, considering that we had not built in lines to indicate inclusion, and that the shapes of the clumps of ideals in Figure 8 are also reverse-engineered. However, these diagrams do give us rough placements of prime ideals which correspond to those in the treasure map, even if our method of drawing inclusion varies. In general, we are led to conclude that placing our ideals roughly vertically by size and imposing various visual cues such as the one proposed in Figure 5, or the one seen in our adaptation of Mumford's treasure map, allows us to encode a decent amount of information in an image, which can combat comprehension barriers.

An Example

In this section, we wish to make it clear that this visualization does not necessarily lend itself to novel solutions to problems. Rather, it is an aid to the visual learner, as it encodes a decent amount of relevant information into a diagram, especially information specific to inclusion.

Problem 1 [Mur20]. Let R be a ring, and consider the two prime ideals P and Q. Show that if there is no prime ideal contained in both P and Q, then there exist open sets U and V in $\operatorname{Spec}(R)$ such that $U \ni P$, $V \ni Q$, and $U \cap V = \varnothing$.

Proof. Let $\mathfrak{a}$ be the ideal such that $U = \boldsymbol{V}(\mathfrak{a})^c$. Then $U \ni P$ is equivalent to $\mathfrak{a} \not\subset P$. Thus if we manage to define U

so that it satisfies the conditions, there will be some $i \in \mathfrak{a} - P$, as in Figure 10. Note that we have not placed any elements other than 0 in their intersection or in $P - \mathfrak{a}$, and we are not implying the existence of such elements with this diagram. A similar diagram could be used to describe a j in $Q - \mathfrak{b}$ where $V = \boldsymbol{V}(\mathfrak{b})^c$.

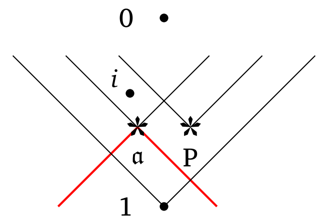

Figure 10: Visualizing $U \ni P$. In general, the elements of U are the primes lying above the red line.

Now visualizing $U \cap V = \varnothing$ is a bit more difficult. Of course, if we find U and V which satisfy all 3 conditions, then if we find $U' \subset U$ and $V' \subset V$ such that $U' \ni P$ and $V' \ni Q$, these open sets will suffice and it may be easier to show that $U' \cap V' = \varnothing$. Looking at Figure 10, we note that $\langle i \rangle \not\subset P$ so it may be easier to find $\langle i \rangle$ and $\langle j \rangle$. We see that we need to find $i \notin P$ and $j \notin Q$ such that every prime ideal contains one of i, j, which is to say that every prime ideal contains ij. Clearly then we must have $i \in Q$ and $j \in P$ so that $ij \in P, Q$, and we wish to find i, j such that in Figure 11, every prime ideal lies below the red line (note that no prime ideals can lie inside the red square, as if a prime contains ij then it must contain one of i, j).

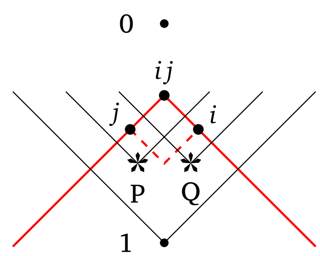

Figure 11: It suffices to find i, j such that every prime ideal lies below the red line.

Now this gives us immediately that it suffices to have ij in the intersection of all prime ideals, namely the nilradical. From here, we only need to use contradiction. If the problem is false, then for any $i \notin P$ and $j \notin Q$ we have ij not in the nilradical, so $(R-P)(R-Q)$ is disjoint from the nilradical. Then by Zorn's lemma there is some ideal maximal among those disjoint from $(R-P)(R-Q)$. Such an ideal must be contained in P and Q so it suffices to show that this ideal is prime, as this would give us a contradiction of the given claim that there is no prime contained in P and Q. This is a common calculation. Let M be our ideal, and say that $a, b \notin M$. Then $\langle a \rangle + M$ contains an element of our multiplicative set since M is maximal among the ideals disjoint from it, so we can write $a = m + ra$ and $b = n + sb$ for $m, n \in M$ and $r, s \in R - M$. Then their product is an element of M plus $rsab$, which is not in M, so M is prime. □

References

[LBr08] Le Bruyn, L. (2008). Mumford's treasure map. neverendingbooks.
`http://www.neverendingbooks.org/mumfords-treasure-map`.

[Mum99] Mumford, D. (1999). The red book of varieties and schemes. Berlin: Springer.

[Mur20] Murayama, T. (2020). MAT447: Homework 3. Princeton University: unpublished assignment.